Constructor University — April 27, 2023
Spring 2023

Topology and Manifolds

Homework 11

Due on May 4, 2023, before class

Problem 1 [**8 points**]

The Brouwer fixed-point theorem states that every continuous function $f : \overline{B}^n \to \overline{B}^n$ has a fixed point (i.e., $f(x) = x$ for at least one x), where $\overline{B}^n$ is the closed n-dimensional unit ball. Prove this theorem using Stokes' theorem.

Hint: First, use a simple picture to show that if f has no fixed point, then there exists a retraction: a continuous function $F : \overline{B}^n \to S^{n-1}$ (where $S^{n-1} = \partial\overline{B}^n$) with every $x \in S^{n-1}$ a fixed point. Then, use Stokes' theorem to prove that such a retraction cannot exist.

Problem 2 [**8 points**]

We consider the upper hemisphere $M := \{x^2 + y^2 + z^2 \le 1, z \ge 0\}$ as a manifold with boundary. Let ω be the 2-form that is given in standard coordinates by

$$\omega = x^3 dy \wedge dz - y^3 dx \wedge dz + z^3 dx \wedge dy.$$

Compute $\int_{\partial M} \omega$ in two ways:

(a) By using Stokes' theorem.

(b) Directly from the definition of the integral. (Note that the trigonometric integrals in the end are a bit nasty, feel free to just look them up.)

(Hint: Remember spherical coordinates.)

Problem 3 [**4 points**]

Let V be a finite-dimensional vector space, and ω a 2-covector on V. We call ω *nondegenerate* if $\hat{\omega} : V \to V^*$, defined by $\hat{\omega}(v) = v \lrcorner \omega$, is invertible. Prove that this is equivalent to:

(a) For each $v \in V$, $v \neq 0$, there exists $w \in V$ such that $\omega(v, w) \neq 0$.

(b) In terms of some (hence every) basis, the matrix ω_{ij} of ω is non-singular.