Prof. Sören Petrat, Office: 112, Research I

Organization:

- See website
- Class: Thu, 9:45, WH-4, shifted to Tue 9:45, WH-4
  Fri, 11:15, WH-4

- Weekly homework
  ↳ see website
  ↳ due a week later before class
  ↳ 2 worst sheets do not count for grading
  ⇒ no late hand-ins, no excuses (except illness longer than a week)

- TA: Dmytro Rudenko
  ↳ HW grading
  ↳ tutorial/office hour announced soon

- Grade: 100% final exam, up to 10% bonus from HW

- Main reference: "Lee - Introduction to Smooth Manifolds"

0. Overview / Motivation

Topology: just short introduction

↳ definitions, basic notions and properties, examples

↳ basic mathematical structure to define convergence, continuity, connectedness, compactness

(Smooth) Manifolds:

- generalizations of curves and surfaces / or: solutions to systems of equations
- things that "locally look like $\mathbb{R}^n$", but not necessarily globally (e.g., sphere, torus)
- but need not be embedded in some $\mathbb{R}^m$
- on basic level: topological manifolds
- more interesting: calculus (curvature, volume) => need extra "smooth structure"
- differential geometry: more extra structure

  (e.g., inner product => Riemannian, Lorentzian, symplectic manifolds)

  Riemannian ↓ distance, angles; Lorentzian ↓ spacetime; symplectic ↓ phase space in classical mechanics

  ↳ only briefly touched here, we rather provide general framework

Structure of this class:

- review of calculus in $\mathbb{R}^n$ (+ Linear algebra)
- topology
- rigorous def. of smooth manifolds and smooth maps + all related aspects
  (tangent space, group actions, submanifolds, rank thm., embeddings, vector fields, tangent/cotangent/vector bundles)
- differential forms
- integration, Stokes thm.
- Lie groups

1. Review of Differentiation in $\mathbb{R}^n$

Derivative = local approximation by linear map

Recall some notation:

- vector $x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \in \mathbb{R}^n$
- Scalar product $\langle x, y \rangle = \sum_{i=1}^{n} x^i \cdot y^i$
- norm: $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{(x^1)^2 + (x^2)^2 + \ldots + (x^n)^2}$
- Landau notation: a fct. $f: \mathbb{R}^m \to \mathbb{R}^n$ is
  - $o(h)$ ("little o") if $\frac{\|f(h)\|}{\|h\|} \xrightarrow{h \to 0} 0$
  - $O(h)$ ("big O") if $\limsup_{h \to 0} \frac{\|f(h)\|}{\|h\|} < \infty$

  (note: sometimes $h \to a \in \mathbb{R}^m$ or $h \to \infty$ is used)
- a map $L: \mathbb{R}^m \to \mathbb{R}^n$ is linear if $L(\lambda x + y) = \lambda L(x) + L(y) \quad \forall \lambda \in \mathbb{R} \ \forall x, y \in \mathbb{R}^m$
- linear map + basis choice = matrix

  let $\{b_i\}_{i=1,\ldots,m}$ be a basis of $\mathbb{R}^m$, e.g., $b_i = e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← i-th position (canonical basis of $\mathbb{R}^m$).

  Then $L(e_1), \ldots, L(e_m)$ uniquely determine the linear map

  $\downarrow \qquad \downarrow$

  column vectors of associated matrix

Next: Differentiability

<u>Def.</u>:

Let $U \subset \mathbb{R}^m$ be open. Then $f: U \to \mathbb{R}^n$ is called **differentiable** at $a \in U$ if there exists a linear map $L: \mathbb{R}^m \to \mathbb{R}^n$ s.t.

$$f(x+h) = f(x) + Lh + o(h) \quad \forall h \in \mathbb{R}^m.$$

L = (total) **derivative** $= Df|_a = Df(a) = f'(a)$ (note: unique)

Ex.: • $f:\mathbb{R}^n \to \mathbb{R}$, $f(x)=\|x\|^2$

$$\Rightarrow f(x+h)=\|x+h\|^2=\|x\|^2+2\langle x,h\rangle+\|h\|^2$$

$$\Rightarrow Df(x)h=2\langle x,h\rangle$$

in canonical basis: $Df(x)=2(x^1,\dots,x^n)$

$$\Rightarrow Df(x)h=2(x^1,\dots,x^n)\begin{pmatrix}h^1\\ \vdots\\ h^n\end{pmatrix}=2\langle x,h\rangle$$

Thm. (chain rule):

let $f:U\to V$ be differentiable at $x\in U$, $g:V\to W$ differentiable at $f(x)\in V$. Then $g\circ f$ is differentiable at x and $D(g\circ f)(x)=Dg(f(x))\circ Df(x)$.

Ex.: $f:\mathbb{R}^n\to\mathbb{R}$, $f(x)=\|x\|^2$, $g:\mathbb{R}\to\mathbb{R}$, $g(y)=e^y$ $\Rightarrow (g\circ f)(x)=g(f(x))=e^{\|x\|^2}$

$\Rightarrow Df(x)=2(x^1,\dots,x^n)$, $Dg(y)=e^y$ $\Rightarrow D(g\circ f)(x)=e^{\|x\|^2}2(x^1,\dots,x^n)$

$(D(g\circ f)(x)h=2e^{\|x\|^2}\langle x,h\rangle)$