Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 2, Feb. 3, 2023

Prof. Dr. Soeren Petrat

Def.:

For $f: \mathbb{R}^n \to \mathbb{R}^n$, we def. the partial derivative $\frac{\partial f}{\partial x^j} := \begin{pmatrix} \frac{\partial f^1}{\partial x_j} \\ \vdots \\ \frac{\partial f^n}{\partial x_j} \end{pmatrix}$

with $\frac{\partial f^i}{\partial x^j}(a) = \lim_{t \to 0} \frac{f^i(a+te^j) - f^i(a)}{t}$. The matrix $\frac{\partial f^i}{\partial x^j}(a)$ called Jacobian matrix at a.

Recall: $f: U \to \mathbb{R}^n$ ($U \subset \mathbb{R}^m$) differentiable at $a \in U \Rightarrow \frac{\partial f^i}{\partial x^j}$ exists for all j and $(Df(a))_{ij} = \sum_{j=1}^{m} \frac{\partial f^i}{\partial x^j}(a)$

Converse?

Ex.: $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

$\Rightarrow \frac{\partial f}{\partial x}((0,0)) = 0 = \frac{\partial f}{\partial y}((0,0))$ but f not continuous at $(0,0)$ (thus also not differentiable at $(0,0)$)

Recall:

Def.: $f: U \to \mathbb{R}^n$ ($U \subset \mathbb{R}^m$) with all partial derivatives continuous on $U \Rightarrow f \in C^1$ ("f is of class C^1")

Thm.: $f \in C^1 \Rightarrow f$ differentiable on U ($f \in C^1$ on $U \Leftrightarrow f$ continuously differentiable on U)

Def.: • $f \in C^k$: all (also mixed) partial derivatives of order k exist and are continuous

- $f \in C^0$: f cont.
- $f \in C^\infty$ or **f smooth** means $f \in C^k \ \forall k \geq 0$
- $f: U \to V$ **diffeomorphism**: smooth + smooth inverse (C^k-diffeomorphism: $C^k + C^k$ inverse)

$U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^n$ (U, V open)

Thm. (Schwarz): $f \in C^2 \Rightarrow \frac{\partial^2 f^i}{\partial x^j \partial x^k} = \frac{\partial^2 f^i}{\partial x^k \partial x^j}$ ($f \in C^k \Rightarrow$ all partial derivatives up to order k commute)

Def.: **directional derivative** of $f: \mathbb{R}^n \to \mathbb{R}$ in direction $v \in \mathbb{R}^n$ at $a \in \mathbb{R}$ is

$$D_v f(a) = \frac{d}{dt} f(a+tv)\Big|_{t=0}.$$

note: • $D_v f(a) \underset{\text{chain rule}}{=} Df(a)v = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a) v^i = \langle \nabla f, v \rangle$

- linear: $D_v(\lambda f + g)(a) = \lambda D_v f(a) + D_v g(a) \quad (\lambda \in \mathbb{R})$
- product rule: $D_v(f \cdot g)(a) = (D_v f(a)) g(a) + f(a)(D_v g(a))$

Recall that an important result is:

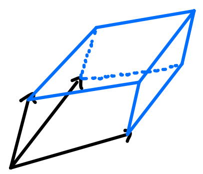
Thm. (Inverse Fct. Thm.):

Let $f: U \to \mathbb{R}^n$ ($U \subset \mathbb{R}^n$ open) be C^k with $Df(a)$ invertible for some $a \in U$. Then $\exists\, V \subset U$ open ($a \in V$), s.t. $f|_V$ has inverse of class C^k and $W = f(V)$ is open. Moreover, $(Df^{-1})(f(x)) = (Df(x))^{-1}$ $\forall x \in V$ (derivative of inverse = inverse of derivative).

Note: If $Df(a)$ not invertible, then a is called **critical point** and $f(a)$ a **critical value**.

Recall: • matrix A invertible (or non-singular) $\Longleftrightarrow \det A \neq 0$

• think of $\det A$ = volume of parallelepiped spanned by column (or row) vectors



volume $= 0 \Longleftrightarrow$ row vector linearly dependent

$\Longleftrightarrow Ax = y$ does not have unique solution x for all y

$\Longleftrightarrow A^{-1}$ does not exist

• det def. by Leibniz or Laplace formula

• $\det(A \cdot B) = \det A \det B \quad \left(\Rightarrow 1 = \det A^{-1}A = \det A^{-1} \det A \Rightarrow \det A^{-1} = \frac{1}{\det A} \right)$

<u>Ex.:</u> $f(x) = \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix}$

$$\Rightarrow Df(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

$\Rightarrow \det Df(x,y) = e^{2x} \cos^2 y + e^{2x} \sin^2 y = e^{2x} > 0 \quad \forall (x,y) \Rightarrow Df$ everywhere non-singular (no critical points)

$\Rightarrow$ inverse fct. thm. applies

but note that f is not globally invertible (periodic in y!)

2. Topology

We put our first structure on a set X.

Def.:
Let X be a set, $\tau = \{U_i \subset X\}_{i \in I}$ (I some index set) with

- $\emptyset, X \in \tau$,
- arbitrary unions of U_i's $\in \tau$,
- finite intersections of U_i's $\in \tau$.

Then each U_i is called open set, each $U_i^c = X \setminus U_i$ closed set, τ a topology, (X, τ) a topological space, any $U_i \ni p$ a (open) neighborhood of p.

Ex.: metric topology on a metric space (X, d)
↳ def. open balls $B_r(x) = \{y \in X : d(x,y) < r\}$ as open
↳ $U \subset X$ is open if $\forall x \in U \ \exists r > 0$ with $B_r(x) \subset U$

[metric d: • $d(x,y) \geq 0$
• $d(x,y) = 0 \iff x = y$
• $d(x,y) = d(y,x)$
• $d(x,z) \leq d(x,y) + d(y,z)$]

More extreme examples:

- Discrete topology: Every subset of X is defined as open.
- Trivial topology: Only $\emptyset$ and X are defined as open.

A topology allows us to define:

- convergence: $(x_i)_{i \geq 0} \to x$ if for every neighborhood U of x $\exists N \in \mathbb{N}$ s.t. $x_i \in U \ \forall i \geq N$
- continuity: preimages of open sets are open

Def.: A bijection $f: X \to Y$ with f and f^{-1} continuous is called homeomorphism.

We often want to study topologies with more structure

Def.:

(X, τ) is called Hausdorff if for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there are (open) neighborhoods U_1 of x_1, U_2 of x_2 with $U_1 \cap U_2 = \emptyset$.

Ex.: • metric topology is Hausdorff (choose $x_1, x_2 \in X$, $\delta = d(x_1, x_2) \Rightarrow U_1 = B_{\delta/3}(x_1), U_2 = B_{\frac{\delta}{3}}(x_2)$)

• Zariski (cofinite) topology on $\mathbb{R}$ (or $\mathbb{C}$): U open $\Leftrightarrow U = \emptyset$ or $X \setminus U$ is finite

↳ not Hausdorff