Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 3, Feb. 7, 2023

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## Generating topologies, basis:

**Def.:** Take any set $X$ and $\mathcal{B}$ a collection of subsets of $X$ with

(a) $X = \bigcup_{B \in \mathcal{B}} B$,

(b) $\forall B_1, B_2 \in \mathcal{B}$, $x \in B_1 \cap B_2$ $\exists B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

Then set of all unions of elements of $\mathcal{B}$ is called the topology generated by $\mathcal{B}$.

note:
- $\emptyset$ is taken to be included in (b)
- it is indeed a topology by def. and due to (b) (finite intersections included)
- alternatively to (b) we could just include finite intersections

Ex.: open balls in $\mathbb{R}^n$ generate standard topology

**Def.:** A collection $\mathcal{B} = \{\text{open sets of } X\}$ is a basis for $(X, \tau)$ if every open subset of $X$ is the union of elements from $\mathcal{B}$.

**Def.:** $(X, \tau)$ is called second-countable if there is a countable basis for $\tau$.

Is $\mathbb{R}^n$ second-countable? Yes, take balls at rational points with rational radius

**Def.:** A collection of (open) subsets of $X$ s.t. their union is $X$ is called (open) cover.

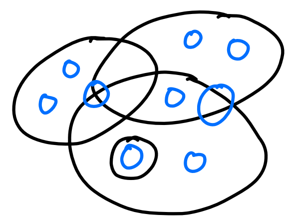
(For $S \subset X$, an open cover of $S$ is a collection of open sets $\{U_i\}_{i \in I}$ s.t. $S \subset \bigcup_{i \in I} U_i$, $I$ some index set.)

A subcollection that is still a cover is called subcover.

**Thm.:** Let $(X,\tau)$ be second-countable. Then every open cover of $X$ has a countable subcover ($=$ Lindelöf space).

**Proof:** Idea: we index some sets of the open cover by basis elements s.t. we still have a subcover

- $\mathcal{B}$ countable basis
- $\mathcal{U}$ some open cover



- $\mathcal{B}_{\mathcal{U}} = \{B \in \mathcal{B} : B \subset V \text{ for some } V \in \mathcal{U}\}$
- for each $B \in \mathcal{B}_{\mathcal{U}}$ choose one $V_B \in \mathcal{U}$, s.t. $B \subset V_B$

$\Rightarrow$ $\mathcal{U}_c = \{V_B : B \in \mathcal{B}_{\mathcal{U}}\}$ is countable; does it still cover $X$?

pick some $y \in X$ (to show: $y \in V_B$ for some $V_B \in \mathcal{U}_c$)

↳ there is $V \in \mathcal{U}$, $y \in V$ ($\mathcal{U}$ open cover)

↳ there is $B \in \mathcal{B}$, $y \in B \subset V$ ($\mathcal{B}$ basis)

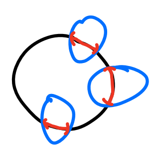
$\Rightarrow B \in \mathcal{B}_{\mathcal{U}} \Rightarrow y \in B \subset V_B$ for some $V_B \in \mathcal{U}_c \Rightarrow \mathcal{U}_c$ open cover $\square$

Next: subspaces and products

**Def.:** $(X,\tau)$ top. space, $S \subset X$. Then the $\boxed{\text{subspace topology}}$ on $S$ is

$$\tau_S = \{U \subset S : U = V \cap S \text{ for some } V \in \tau\}$$

Ex.: natural top. on a circle



**Def.:** Let $(X_1,\tau_1),\dots,(X_k,\tau_k)$ be top. spaces. The $\boxed{\text{product topology}}$ on $\overset{\text{Cartesian product}}{\overset{\downarrow}{X_1\times X_2\times\dots X_k}}$ is the top. generated by $\{U_1\times\dots\times U_k : U_i\in\tau_i, i=1,\dots,k\}$, the corresponding top. space is called $\boxed{\text{product space}}$.

**Def.:** $\pi_i : X_1\times\dots\times X_k\to X_i$, $\pi_i(x_1,\dots,x_k)=x_i$ is called i-th $\boxed{\text{canonical projection}}$.

note:
- $\pi_i$'s are continuous: $\pi_i^{-1}(U_i)=X_1\times\dots\times U_i\times\dots\times X_k$
- $f: Y\to X_1\times\dots\times X_k$ cont. $\Longleftrightarrow$ $f_i=\pi_i\circ f: Y\to X_i$ (component fct.s) cont.

  ("$\Rightarrow$" composition of cont. fct.s; "$\Leftarrow$" by def.)

Next: compactness: "finiteness conclusions on infinite sets"

**Def.:** A top. space $(X,\tau)$ is $\boxed{\text{compact}}$ if every open cover of $X$ has a finite subcover.

note: $\boxed{\text{compact subset}}$ means it is compact in subspace topology

recall main results from Analysis:
- If $(X,\tau)$ compact, then every continuous $f: X\to\mathbb{R}$ assumes its maximum and minimum.
- Heine-Borel: $X\subset\mathbb{R}$ compact $\Leftrightarrow$ $X$ closed and bounded
- $f: X\to Y$ cont., $X$ compact $\Rightarrow$ $f(X)$ compact
- $f: M_1\to M_2$, $(M_1,d_1)$ and $(M_2,d_2)$ metric spaces, $K\subset M_1$ compact. Then $f$ cont. $\Rightarrow$ $f|_K$ uniformly cont.

Note: $X$ second-countable Hausdorff or metric space:

$X$ compact $\iff$ every sequence in $X$ has a convergent subsequence with limit in $X$

(= sequential compactness)

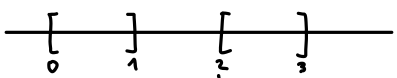
Next: (path-) connectedness

Def.: A top. space $(X,\tau)$ is connected if the only subsets of $X$ that are both open and closed are $X$ and $\emptyset$.

note: $(X,\tau)$ disconnected $\iff \exists\, U, V$ non-empty, disjoint and open s.t. $X = U \cup V$

("$\Rightarrow$" $\exists\, U$ open and closed, $U \neq X, U \neq \emptyset \Rightarrow U^c$ open and closed $\Rightarrow U \cup U^c = X$

"$\Leftarrow$" $U^c = V$ open and closed and neither $= \emptyset$ nor $= X$ )

Ex.: $X = [0,1] \cup [2,3]$ with subspace topology 

E.g., $\underbrace{B_1(\tfrac{1}{2})}_{=(-\frac{1}{2},\frac{3}{2})} \cap \big([0,1] \cup [2,3]\big) = [0,1]$, so $[0,1]$ is open, but also closed $\Rightarrow X$ disconnected