Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 4, Feb. 10, 2023

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Recall

Def.: A top. space (X,τ) is connected if the only subsets of X that are both open and closed are X and $\emptyset$.

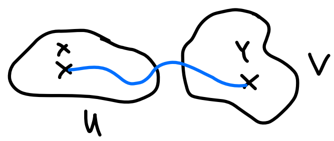
Note: · For $S \subset X$, the boundary of S is def. as $\partial S = \{p \in X :$ all neighborhoods of p have at least one point in S and one not in $S\}$

· S both open and closed $\iff \partial S = \emptyset$

Def.: A top. space (X,τ) is path-connected if $\forall x,y \in X$ $\exists$ cont. $\gamma : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$ (γ = path)

Note: · path conn. $\Rightarrow$ conn.

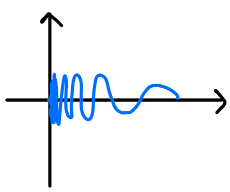
(proof:



path has to cross boundary, but sets that are both open and closed have no boundary)

· For open subsets of $\mathbb{R}^n$ (or $\mathbb{C}^n$): path conn. $\iff$ conn.

· Example for X that is conn. but not path conn.: topologist's sine curve $\{(x, \sin\frac{1}{x}) : x \in (0,1]\} \cup \{(0,0)\}$



$\to$ topologically connected, but no path from 0 to rest of curve

· A maximal connected subset of X is called connected component of X

- $I \subset \mathbb{R}$ connected $\Longleftrightarrow$ I interval or point
- $f: X \to Y$ cont., X (path-) connected $\Longrightarrow$ $f(X)$ (path-)connected

  ($f(X)$ not conn. $\Rightarrow$ $\exists\, V \subset f(X)$ open and closed and $\neq \emptyset$, $\neq f(X)$ $\Rightarrow$ same for $f^{-1}(V)$ by cont.
  $\Rightarrow$ contradiction)

- $f: X \to \mathbb{R}$ cont., X conn., suppose $\exists\, a, b \in X$ s.t. $f(a) < 0 < f(b)$

  $\Rightarrow \exists\, c \in X$ s.t. $f(c) = 0$ (X conn. $\Rightarrow f(X)$ conn. $\Rightarrow f(X) =$ interval)

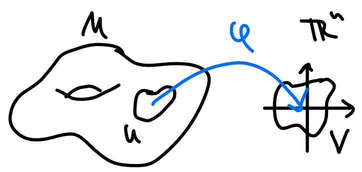
2. Manifolds: Definition and Examples

2.1 Topological Manifolds

Manifold: "looks locally like $\mathbb{R}^n$"

Def.: A topological manifold M of dimension n is a Hausdorff and second-countable topological space s.t. every point in M has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$.
($\forall p \in M$ $\exists$ open $U \subset M$, $p \in U$, and open $V \subset \mathbb{R}^n$, and homeomorphism $\varphi : U \to V$.)

note: • equivalently: could use homeomorphic to some open ball in $\mathbb{R}^n$ (or even unit ball in $\mathbb{R}^n$)
Why? Let $\varphi: U \to V$ as above $\Rightarrow \exists\, r > 0$ s.t. $B_r(\varphi(p)) \subset \Phi(U)$



$\Rightarrow$ use $\varphi : \varphi^{-1}(B_r(\varphi(p))) \longrightarrow B_r(\varphi(p))$
(unit ball by rescaling)

• the dimension of a manifold is a topological invariant: an n-dim. manifold is never homeomorphic to an m-dim manifold for $m \neq n$

Def.: A pair (U, φ) with $U \subset M$ open, homeomorphism $\varphi : U \to V$ for open $V = \varphi(U) \subset \mathbb{R}^n$ is called (coordinate) chart. Also, we call:

- φ a (local) coordinate map,
- $\varphi(p) = (x^1(p), \dots, x^n(p))$ local coordinates,
- $\varphi^{-1} : V \to U$ a coordinate system.

Examples: • any open subset of $\mathbb{R}^n$ is a top. n-manifold

• n-sphere $S^n := \{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} x_j^2 = 1\}$

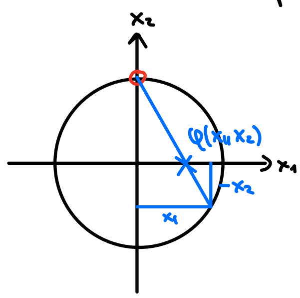
↳ Hausdorff and second countable clear

(note: Hausdorff and second countability generally transfer to subsets with subspace top.)

↳ charts: e.g., use stereographic projections

$$\varphi^+ : S^n \setminus \underbrace{\{(0,\dots,0,1)\}}_{\text{north pole}} \to \mathbb{R}^n, \quad \varphi^+(x^1,\dots,x^{n+1}) = \frac{1}{1-x^{n+1}}(x^1,\dots,x^n)$$

$$\varphi^- : S^n \setminus \underbrace{\{(0,\dots,0,-1)\}}_{\text{south pole}} \to \mathbb{R}^n, \quad \varphi^-(x^1,\dots,x^{n+1}) = \frac{1}{1+x^{n+1}}(x^1,\dots,x^n)$$



$$\varphi(x_1,x_2) = \frac{x_1}{1-x_2}$$

$\varphi^{\pm}$ are both homeomorphisms and their domains cover S^n

$\Rightarrow S^n$ is top. n-manifold

<u>Thm.:</u> Let $M_1,\dots,M_k$ be top. manifolds of dim. $n_1,\dots,n_k$. Then $M_1\times\dots\times M_k$ is a top. manifold of dim. $n_1+\dots+n_k$.

<u>Proof:</u> Hausdorff and second-countable follows directly for product topology.

Locally like $\mathbb{R}^n$: For each $(p_1,\dots,p_k) \in M_1\times\dots\times M_k$ choose corresponding charts (U_i,φ_i)

$\Rightarrow \varphi_1\times\dots\times\varphi_k : U_1\times\dots\times U_k \to \mathbb{R}^{n_1+\dots+n_k}$ is homeomorphism onto its image □

Ex.: n-torus $T^n = S^1\times\dots\times S^1$