Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 5, Feb. 16, 2023

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Def.: A topological space is called locally path-connected if it has a basis of path-connected open subsets.

Thm.: For a locally path-connected top. space (X,τ):

- All connected components are open.
- X connected $\Leftrightarrow$ X path connected.

Note: all conn. comp. closed clear. Finitely many $\Rightarrow$ all open, but e.g. rationals: singletons are conn. components, but they are not open

C connected $\Rightarrow$ $\bar{C}$ connected

Proof: HW

For top. manifolds this implies:

Thm.: Let M be a top. manifold. Then:

a) M is locally path-connected.

b) M connected $\Leftrightarrow$ M path-connected.

c) M has countably many connected components, each of which is open and thus a top. manifold.

Proof: a) M has a basis of coordinate balls by definition.

b) Follows from a) and previous thm.

c) Each component is open due to previous thm. $\Rightarrow$ collection of components is an open cover (of disjoint sets) $\Rightarrow$ countably many bc. M is second countable. $\square$

Another example of a manifold: n-dim. real projective space $\mathbb{P}^n$

↳ recall: $\sim$ is an equivalence relation if: $\cdot\ x \sim x$ (reflexive)

$\cdot\ x \sim y \Rightarrow y \sim x$ (symmetric)

$\cdot\ x \sim y$ and $y \sim z \Rightarrow x \sim z$ (transitive)

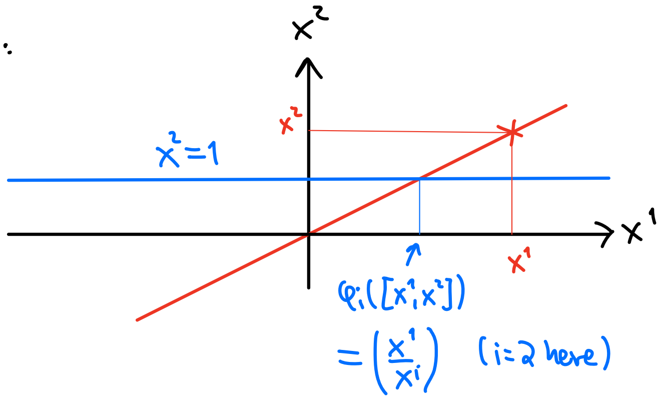
equivalence class $[x] := \{y : x \sim y\}$

↳ here we def. for $x, y \in \mathbb{R}^{n+1}$ that $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{R}$

$\Rightarrow \mathbb{P}^n$ = set of all equivalence classes (= all straight lines through origin = all 1-dim. linear subspaces of $\mathbb{R}^n$)

Def. natural map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, $\pi(x) = [x]$.

Construction of a chart:



let $\hat{U}_i = \{y \in \mathbb{R}^{n+1} \setminus \{0\} : x^i \neq 0\}$ and $U_i = \pi(\hat{U}_i)$

unchanged if x is multiplied by a constant ($\neq 0$).

$\Rightarrow \varphi_i : U_i \to \mathbb{R}^n$, $\varphi_i([x^1, \dots, x^{n+1}]) = \frac{1}{x^i}(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{n+1})$

with $\varphi_i^{-1}(u^1, \dots, u^n) = [u^1, \dots, u^{i-1}, 1, u^i, \dots, u^n]$, both φ_i and φ_i^{-1} are continuous (see HW)

Since $U_1, \dots, U_{n+1}$ cover $\mathbb{P}^n$ (+ Hausdorff + second countable), $\mathbb{P}^n$ is an n-manifold