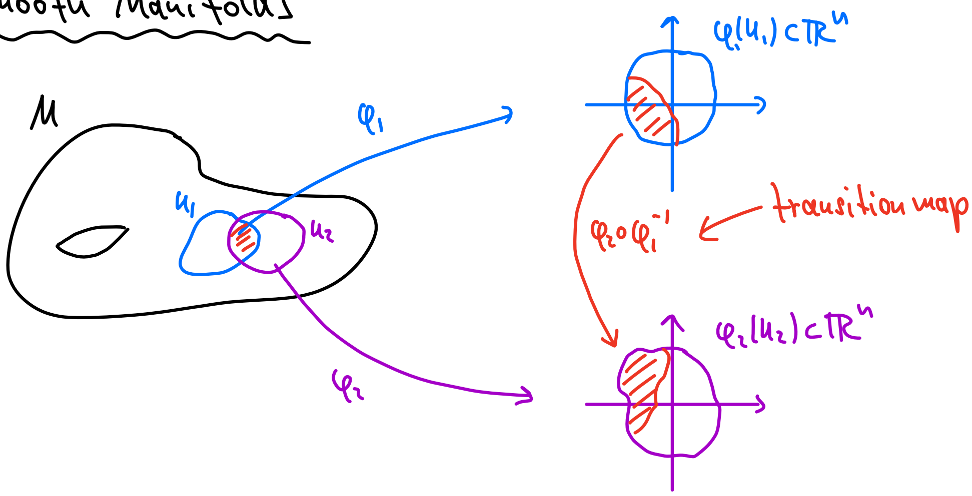
2.2 Smooth Manifolds



For a sensible def. of smooth fct.'s on M, we need a smooth structure on M

Def.: Let M be a top. n-manifold. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ for some index set I, s.t.

- U_α are open and cover M,
- $\forall \alpha, \beta$ with $U_\alpha \cap U_\beta \neq \emptyset$, the **transition map** $\varphi_\beta \circ \varphi_\alpha^{-1} : \underbrace{\varphi_\alpha(U_\alpha \cap U_\beta)}_{\subset \mathbb{R}^n, \text{ open}} \to \underbrace{\varphi_\beta(U_\alpha \cap U_\beta)}_{\subset \mathbb{R}^n, \text{ open}}$ is C^r ($(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are C^r compatible)

Then $\mathcal{A}$ is called a **C^r atlas** for M, and $(M, \mathcal{A})$ a **C^r manifold**.

Note:
- smooth atlas/manifold $= C^\infty$ atlas/manifold
- just atlas means C^0 atlas ($= \{U_\alpha\}$ open cover)
- in general atlas is not unique (starting with a top. manifold)
- any C^k atlas is also a C^ℓ atlas if $k > \ell$
- how to check that (U, φ) and (V, ψ) are C^r compatible (i.e., $\psi \circ \varphi^{-1}$ a C^r diffeomorphism)?
  ↳ check if $\psi \circ \varphi^{-1}$ is C^r and injective and Jacobian non-singular
  $\Rightarrow$ C^r compatible by inverse fct. thm.

Non-uniqueness usually not a problem if we use the following:

<u>Def.:</u> A C^r (differentiable) structure on M is a maximal C^r atlas $\mathcal{A}$ (i.e., not contained in any larger C^r atlas).

Note: • maximal C^r atlas = union of all C^r-equivalent atlases ($\mathcal{A}, \mathcal{A}'$ are C^r equivalent if $\mathcal{A} \cup \mathcal{A}'$ is C^r atlas)

• one can show (simple exercise): any C^r atlas is contained in exactly one maximal C^r atlas (but a top. manifold might have many such maximal C^r atlases)

• furthermore: for every C^r structure $\exists$ unique C^r-equivalent C^∞ structure
$\Rightarrow$ usually we consider smooth manifolds only

Some hard problems: • top. manifold that does not admit any smooth structure (ex. by Kervaire 1960)

• how many smooth structures does n-sphere have?

↳ $n=1,2,3,5,6$: one
↳ $n=4$: unknown
↳ $n=7$: 28 (first ex.: Milnor 1956)
...

Examples:

• $\mathbb{R}^n$ is a smooth n-manifold, e.g., choose one chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$ (Standard smooth structure on $\mathbb{R}^n$.)

Other example: $\{(B_1(x), \mathrm{Id}_{B_1(x)}) : x \in \mathbb{R}^n\}$ (Both are contained in the same smooth structure.)

• On $\mathbb{R}$, $(\mathbb{R}, f)$ with $f(x)=x^3$ defines a smooth structure different from the standard smooth structure ($\mathrm{Id} \circ f^{-1}(y) = y^{\frac{1}{3}}$ not smooth at origin.)

• product smooth manifold structure

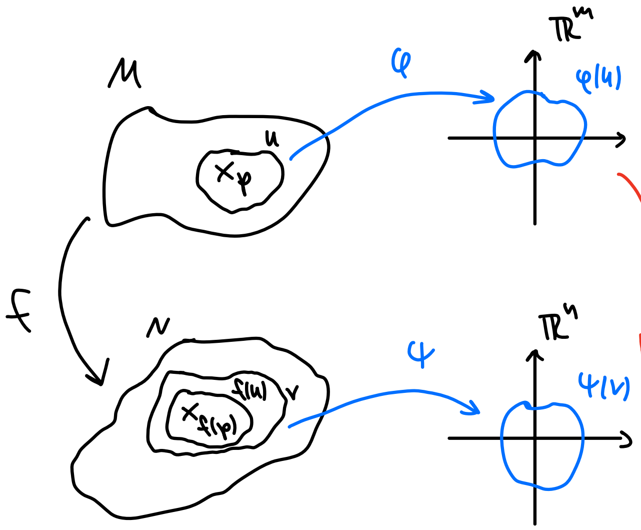
- open subset $U \subset M$ of smooth manifold $(M, \mathcal{A})$

  $\Rightarrow \mathcal{A}_U := \{(V \cap U, \varphi|_{V \cap U}) : (V, \varphi) \in \mathcal{A}\}$, then $(U, \mathcal{A}_U)$ is a smooth manifold

- sphere S^n: use either stereographic projections or direct projections, see HW

- real projective space $\mathbb{RP}^n$: see HW

2.3 Smooth Maps between Manifolds



<u>Def.:</u> Let M, N be smooth manifolds. A $\boxed{\text{map } f: M \to N \text{ is smooth at } p \in M}$ if there are charts (U, φ) with $p \in U$ and (V, ψ) with $f(p) \in V$ s.t. $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1} : \underset{\mathbb{R}^m}{\varphi(U)} \to \underset{\mathbb{R}^n}{\psi(V)}$ is smooth at $\varphi(p)$.

note:
- f smooth $\Leftrightarrow$ f smooth $\forall p \in M$
- $f: M \to \mathbb{R}^k$ smooth $\Rightarrow$ $N = \mathbb{R}^k$, and can choose $V = \mathbb{R}^k$, $\psi = id$ in def.
- smoothness property independent of choice of chart due to def. of smooth atlas
- simple ex.s: identity $id: M \to M$ and constant maps $f: M \to M$ are smooth

Standard results:

<u>Proposition:</u> f smooth $\Rightarrow$ f continuous

<u>Proof:</u> Notation as in def., then $f|_U = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \varphi : U \to V$ is cont. as composition of cont. fct.s $\Rightarrow$ f cont. by HW2 Problem 1. $\square$

Standard statements with straightforward proof:

<u>Proposition:</u>

- $f_i : M \to N_i$ smooth $\Rightarrow$ $f : M \to N_1 \times \cdots \times N_k$, $f(p) = (f_1(p), \ldots, f_k(p))$ smooth
- $f : M \to N$ and $g : N \to P$ smooth $\Rightarrow$ $g \circ f : M \to P$, $(g \circ f)(p) = g(f(p))$ smooth
- $f, g : M \to \mathbb{R}^n$, $\lambda : M \to \mathbb{R}$ smooth $\Rightarrow$ $f + g$, λf, and $\langle f, g \rangle$ smooth

$\langle f, g \rangle : M \to \mathbb{R}$, $\langle f, g \rangle(x) = \langle f(x), g(x) \rangle$
$= \sum_{i=1}^{n} f_i(x) g_i(x)$