Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 7, Feb. 23, 2023

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Classification of smooth manifolds:

Def.: Let M, N be smooth manifolds. A homeomorphism $f: M \to N$ s.t. f and f^{-1} are smooth is called a **diffeomorphism**. M and N are called **diffeomorphic** $(M \approx N)$ if there exist a diffeomorphism $f: M \to N$.

note: • $M \approx M$ (use id.)

• $M \approx N \Rightarrow N \approx M$ (by def.)

• $M \approx N$ (with diffeomorphism f), $N \approx P$ (with diffeom. g) $\Rightarrow M \approx P$ (using $f \circ g$)

$\Rightarrow$ equivalence class of diffeomorphic manifolds

↳ classification of 3-manifolds is a current research direction

↳ ex.: any compact connected 1-manifold is diffeomorphic to S^1

Ex.: • $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$, $\mathbb{R}^n$ with standard smooth structure

↳ take $f: \mathbb{R}^n \to \mathbb{B}^n$, $f(x) = \dfrac{x}{\sqrt{1+\|x\|^2}}$ $\Rightarrow$ smooth

↳ inverse $f^{-1}: \mathbb{B}^n \to \mathbb{R}^n$, $f^{-1}(y) = \dfrac{y}{\sqrt{1-\|y\|^2}}$ $\Rightarrow$ smooth

$\Rightarrow \mathbb{B}^n$ diffeomorphic to $\mathbb{R}^n$

Homeomorphism invariance of dim. requires an advanced proof, but diffeomorphism invariance is easier:

Let $f: M \to N$ be a diffeomorphism $\Rightarrow \psi \circ f \circ \varphi^{-1}: \underbrace{\varphi(U)}_{\subset \mathbb{R}^m} \to \underbrace{\psi(V)}_{\subset \mathbb{R}^n}$ is a diffeomorphism $\Rightarrow m = n$

(see also HW)

Diffeomorphism invariant properties are a central research subject.

2.4 Partitions of Unity

Tool to "glue together" local smooth objects into global ones.

Def.: Let $r_1 < r_2 \in \mathbb{R}$.

- A smooth fct. $h: \mathbb{R} \to \mathbb{R}$ with $h(x) = \begin{cases} 1, & x \leq r_1, \\ \text{between } 0 \text{ and } 1, & r_1 < x < r_2, \\ 0, & x \geq r_2, \end{cases}$ is called **cutoff fct.**

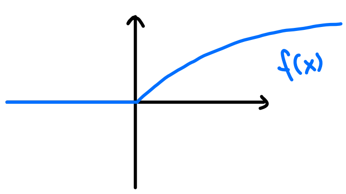
- A smooth fct. $H: \mathbb{R}^n \to \mathbb{R}$ with $H(x) = \begin{cases} 1 & \text{on } \overline{B_{r_1}(0)}, \\ \text{between } 0 \text{ and } 1 \text{ on } & B_{r_2}(0) \setminus \overline{B_{r_1}(0)}, \\ 0 & \text{on } \mathbb{R}^n \setminus B_{r_2}(0), \end{cases}$

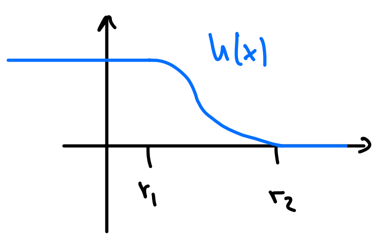
  is called (smooth) **bump fct.**

Such fct.s exist:

E.g., def. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$, from Analysis we know that f is smooth

Then $h(x) = \dfrac{f(r_2 - x)}{f(r_2 - x) + f(x - r_1)}$ is a smooth cutoff fct., $H(x) = h(\|x\|)$ a smooth bump fct.





Note/recall: $\mathrm{supp}(f) := \overline{\{x \in M : f(x) \neq 0\}}$ (support of f)

Next: glueing together for manifolds

Def.: Let $\mathcal{X} = \{X_\alpha\}_{\alpha\in A}$ be an open cover of a manifold M. We call $\{\psi_\alpha : M \to \mathbb{R} \text{ cont. (smooth)}\}_{\alpha\in A}$ a (smooth) partition of unity subordinate to $\mathcal{X}$ if:

- $0 \leq \psi_\alpha(x) \leq 1 \quad \forall \alpha\in A, x\in M$
- $\operatorname{supp} \psi_\alpha \subset X_\alpha \quad \forall \alpha \in A$
- $\{\operatorname{supp} \psi_\alpha\}_{\alpha\in A}$ locally finite (each $x\in M$ has neighborhood U that intersects finitely many $\operatorname{supp} \psi_\alpha$'s)
- $\sum_{\alpha\in A} \psi_\alpha(x) = 1 \quad \forall x\in M$

Thm.: For a smooth manifold M and any open cover $\mathcal{X} = \{X_\alpha\}_{\alpha\in A}$ there exists a smooth partition of unity subordinate to $\mathcal{X}$.

We skip the proof and refer to HW for applications.

2.5 Tangent Space

We first recall some basic Linear Algebra:

Def.: A vector space over $\mathbb{R}$ (or any field $\mathbb{F}$) is a set V with two operations:

- addition: $V \times V \to V$
- scalar multiplication: $\mathbb{R} \times V \to V$

that satisfy:

- V is an abelian group under addition, i.e., associative, commutative, $\exists$ zero, $\exists$ inverse
- $\lambda_1(\lambda_2 v) = (\lambda_1 \lambda_2) v \quad \forall v \in V, \lambda_1, \lambda_2 \in \mathbb{R}$

  $1 \cdot v = v \quad \forall v \in V$
- distributive: $(\lambda_1 + \lambda_2) v = \lambda_1 v + \lambda_2 v \quad \forall v \in V, \lambda_1, \lambda_2 \in \mathbb{R}$

  $\lambda(v+w) = \lambda v + \lambda w \quad \forall v, w \in V, \lambda \in \mathbb{R}$

Ex.s: $\mathbb{R}^n$, $n \times m$ matrices, Polynomials of degree $\leq n$

Recall notions of linear combination, linear (in)dependence, basis

Def.: A map $T: V \to W$, V, W vector spaces is linear if $T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2)$

$\forall v_1, v_2 \in V, \lambda_1, \lambda_2 \in \mathbb{R}$

A bijective linear map is called isomorphism.

Note:
- f isomorphism $\Rightarrow f^{-1}$ linear $\quad \left(f(\lambda v + w) = \lambda f(v) + f(w) \Rightarrow \lambda v + w = f^{-1}(\lambda f(v) + f(w)) \right)$
- V, W isomorphic ($\exists$ isomorphism) $\Leftrightarrow \dim V = \dim W$
- Canonical isomorphisms are those that involve no arbitrary choices (e.g., of basis)
- All norms on finite-dim. vector spaces are equivalent $\left(\|v\|_1 \leq c \|v\|_2 \leq \tilde{c} \|v\|_1 \right)$

  $\Rightarrow$ all lead to same topology

  Given basis $(e_1, \dots, e_n)$, def. isomorphism $E: \mathbb{R}^n \to V$, $E(x) = \sum_{i=1}^n x^i e_i \quad ((V, E^{-1})$ chart$)$

  $\Rightarrow$ topological manifold

If $(\tilde{e}_1, \dots, \tilde{e}_n)$ is another basis, $\tilde{E}(x) = \sum_{i=1}^{n} x^i \tilde{e}_i$, $\exists$ matrix of basis change A_i^j, i.e., $e_i = \sum_{j=1}^{n} A_i^j \tilde{e}_j$

$\Rightarrow$ transition map $\tilde{E}^{-1} \circ E(x) = \tilde{E}^{-1}\left(\sum_{i=1}^{n} x^i e_i\right) = \sum_{i=1}^{n} x^i \underbrace{\tilde{E}^{-1}(e_i)}_{= \tilde{E}^{-1}\left(\sum_{j=1}^{n} A_i^j \tilde{e}_j\right)} = \left(\sum_i A_i^1 x^i, \dots, \sum_i A_i^n x^i\right)$

↳ diffeomorphism

$\Rightarrow$ $\{(V, E^{-1}) : E$ def. via basis$\}$ smooth atlas $\Rightarrow$ smooth manifold

$\Rightarrow$ standard smooth structure