Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 8, Feb. 24, 2023

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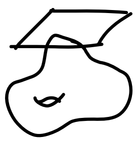
Next: def. derivatives for $f: M \to N$

Ideas: • need linear approximation to f (but manifolds don't have a linear structure)

•  derivative lives in what we will call the tangent space

↳ but: want def. independent of any embedding in higher-dim. space

• also want def. indep. of coordinates

• rough idea: space of all directional derivatives

Recall: directional derivative of $f: \mathbb{R}^n \to \mathbb{R}$ at point a in direction v:

$$D_v|_a f = D_v f(a) = \frac{d}{dt} f(a+tv)\Big|_{t=0} \underset{\text{(in standard basis)}}{=} \sum_i v^i \frac{\partial f}{\partial x^i}(a)$$

↳ product rule: $D_v(fg)(a) = f(a) D_v g(a) + g(a) D_v f(a)$

Def.: For $a \in \mathbb{R}^n$, a map $w: C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called **derivation at a** if it is linear and satisfies Leibniz rule $w(fg) = f(a) wg + g(a) wf$ $\forall f,g \in C^\infty(\mathbb{R}^n)$.

$T_a\mathbb{R}^n := \{w : w \text{ derivation at } a\}$

Note: • $D_v|_a$ is a derivation

• some computations show that $T_a\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ (or $\mathbb{R}^n_a = \{a\} \times \mathbb{R}^n$, the geometric tangent space)

↳ $v \mapsto D_v|_a$ is isomorphism (see next HW)

• $\left(\frac{\partial}{\partial x^1}\Big|_a, \dots, \frac{\partial}{\partial x^n}\Big|_a\right)$ is a basis of $T_a\mathbb{R}^n$ $\left(\frac{\partial}{\partial x^i}\Big|_a f := \frac{\partial f}{\partial x^i}(a)\right)$

On a general manifold M we analogously define:

Def.: Let M be a smooth manifold, $p \in M$. A linear map $D: C^\infty(M) \to \mathbb{R}$ is called **derivation at p** if $D(fg) = f(p) Dg + g(p) Df \quad \forall f,g \in C^\infty(M)$ (smooth $M \to \mathbb{R}$)

$T_pM = \{D : D \text{ derivation}\}$ is called **tangent space** to M at p.

Note: T_pM is a vector space (so now we have a linear structure for our derivatives)

Ex.: • If $f = \text{const} = c$, then $v(f) = c\,v(1) = c\,v(1 \cdot 1) = c\,(1 \cdot v(1) + 1 \cdot v(1)) = 2c\,v(1) = 2v(f)$

$\Rightarrow v(f) = 0 \quad \forall v \in T_pM$

• If $f(p) = g(p) = 0$ then $v(fg) = 0 \quad \forall v \in T_pM$

Def.: Let M, N be smooth manifolds, $F: M \to N$ smooth, $p \in M$. The **differential of F at p** or **push forward** is a map $dF_p : T_pM \to T_{F(p)}N$ def. by

$$\mathbb{R} \ni \{ \boxed{\underbrace{dF_p(\underbrace{v}_{\in T_pM})}_{\in T_{F(p)}N}(\underbrace{f}_{\in C^\infty(N)}) = \underbrace{v(\underbrace{f \circ F}_{M \to \mathbb{R}})}_{\in \mathbb{R}}} \qquad \forall v \in T_pM,\ f \in C^\infty(N).$$

Note: $dF_p(v)$ is indeed linear (v derivation) and a derivation at $F(p)$:

$$dF_p(v)(fg) = v((fg) \circ F) = v((f \circ F) \cdot (g \circ F))$$
$$= (f \circ F)(p)\, v(g \circ F) + (g \circ F)(p)\, v(f \circ F)$$
$$= f(F(p))\, dF_p(v)(g) + g(F(p))\, dF_p(v)(f)$$

Properties: M, N, P smooth manifolds, $F: M \to N$, $G: N \to P$ smooth, $p \in M$, then

(proofs omitted) • $dF_p : T_pM \to T_{F(p)}N$ is linear

• $d(G \circ F)_p : T_pM \to T_{G \circ F(p)}P$, $d(G \circ F)_p = \underbrace{dG_{F(p)}}_{T_{F(p)}N \to T_{G \circ F(p)}P} \circ \underbrace{dF_p}_{T_pM \to T_{F(p)}N}$

see HW

• $d(\overset{\text{identity}}{Id_M})_p = Id_{T_pM}$ ($d(id)_p(v) = v$ by def.)

• If F diffeomorphism $\Rightarrow dF_p$ isomorphism and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$

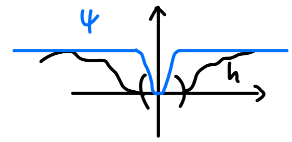
Important: tangent spaces are really local!

<u>Proposition:</u> M smooth manifold, $U \subset M$ open, $i: U \to M$ inclusion map. Then $\forall p \in U$,

$di_p : T_pU \to T_pM$ is a canonical isomorphism.

$\Rightarrow$ $di_p(v)$ and v are really "the same" and we will identify them in the following

Proof uses this Proposition: let $p \in M$, $v \in T_pM$. If for $f,g \in C^\infty(M)$, $f|_U = g|_U$ for some neighborhood U of p then $vf = vg$.



we just prove this: • $h := f - g$, so $h|_U = 0$

• ψ = bump fct. with $\psi = 1$ on $\mathrm{supp}(h)$, $\mathrm{supp}\,\psi \subset M \setminus \{p\}$

$\Rightarrow \psi h = h \Rightarrow h(p) = \psi(p) = 0 \Rightarrow v(h\psi) = h(p)\, v\psi + \psi(p)\, vh = 0$

$\qquad \qquad \| $

$\qquad v(h) \Rightarrow vf = vg$ $\square$

(Then use extension of smooth fct.s)

<u>Corollary:</u> T_pM has same dimension as M.

↳ proven by using a local smooth chart at p together with previous Proposition

How to do computations?

Choose smooth chart (U, φ) at p $\Rightarrow$ $d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ isomorphism

basis $\left.\frac{\partial}{\partial x^1}\right|_{\varphi(p)}, \dots, \left.\frac{\partial}{\partial x^n}\right|_{\varphi(p)}$

$$\Rightarrow \left.\frac{\partial}{\partial x^i}\right|_p = \underbrace{(d\varphi_p)^{-1}}_{=(d\varphi^{-1})_{\varphi(p)}} \left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \right) \quad \text{(coordinate) basis of } T_pM$$

note: $f \in C^\infty(U) \Rightarrow \left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} (f \circ \varphi^{-1})$

$$= \frac{\partial \hat{f}}{\partial x^i}(\hat{p})$$

(recall def. $dF_p(v)(f) = v(f \circ F)$)

$\hat{f} = f \circ \varphi^{-1}$, $\hat{p} = \varphi(p)$ coordinate representations of f and p

So what is dF_p in local coordinates (for $F: M \to N$)? $\longrightarrow$ see HW