Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 9, Mar. 2, 2023
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3. Embeddings, Submanifolds, Sard's Theorem

3.1 Local Structure of Maps between Manifolds

Recall from linear Algebra:

- linear map $T: V \to W$, $\operatorname{im}(T) = \{Tv \in W : v \in V\}$, $\operatorname{rank}(T) = \dim(\operatorname{im} T)$
  $\ker(T) = \{v \in V : Tv = 0\}$, $\operatorname{nullity}(T) = \dim(\ker T)$
- for any linear map T of rank r one can choose bases (of V and W) s.t. matrix of $T = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

  $\Rightarrow \dim V = \dim(\operatorname{im} T) + \dim(\ker T)$

  $\Rightarrow \operatorname{rank}(T)$ is def. indep. of choice of basis (the only basis-indep. (if bases of V and W chosen indep.) property of a general lin. map T)
- T injective $\Leftrightarrow \ker T = \{0\} \Leftrightarrow \dim(\operatorname{im} T) = \dim V \Leftrightarrow T = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$ in some basis
- T surjective $\Leftrightarrow \dim(\operatorname{im} T) = \dim W \Leftrightarrow T = (I_r, 0)$ in some basis

Back to manifolds:

Def.: let M, N be smooth manifolds, $F: M \to N$ smooth.
If $\operatorname{rank} dF_p = r$ $\forall p \in M$, we say F has **constant rank** ($\operatorname{rank} F = r$).
Furthermore, we call F

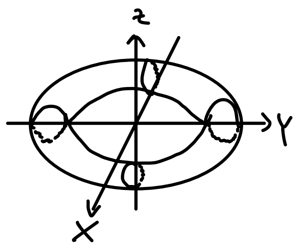
- **submersion** if dF_p is surjective ($\operatorname{rank} dF_p = \dim N$) $\forall p \in M$

important for discussing submanifolds:
- **immersion** if dF_p is injective ($\operatorname{rank} dF_p = \dim M$) $\forall p \in M$
- **embedding** if F is an immersion and $F: M \to F(M)$ a homeomorphism.

Ex.s:

- $F:\mathbb{R}^2\to\mathbb{R}$, $F(x,y)=x$ $\Rightarrow \left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y}\right)=(1,0) \Rightarrow \operatorname{rank} dF_p=1$ $\forall p\in\mathbb{R}^2 \Rightarrow F$ submersion
- $F:\mathbb{R}^2\to\mathbb{R}$, $F(x,y)=x^2+y^2 \Rightarrow \left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y}\right)=2(x,y) \Rightarrow \operatorname{rank} dF_{(0,0)}=0 \Rightarrow F$ not a submersion (not even const. rank)
- projections $\pi_i: M_1\times\dots\times M_k\to M_i$ are submersions $\left(d\pi_i|_p(v)(f)=v(f\circ\pi_i)\right)$
- smooth curves $\gamma:(-1,1)\to M$ with $\gamma'(t)\neq 0$ $\forall t\in(-1,1)$ are immersions $\left(d\gamma_t\left(\frac{\partial}{\partial t}\right)(f)=\frac{\partial}{\partial t}(f\circ\gamma)\right)$

  (e.g., $\gamma(t)=(t^3,0)$ is not an immersion since $\gamma'(0)=(0,0)$)
- $U\subset M$ open, inclusion map $i:U\to M$ is an embedding
- $F:\mathbb{R}^2\to\mathbb{R}^3$, $F(u,v)=\left((2+\cos 2\pi u)\cos(2\pi v),(2+\cos 2\pi u)\sin(2\pi v),\sin(2\pi u)\right)$ is an immersion



With some work we can show that $F:S^1\times S^1\to\mathbb{R}^3$ is also an embedding.

Next: consider such submanifolds (like torus $=\operatorname{Im}F$, which is not open in $\mathbb{R}^3$)

Important result:

Rank-thm.: If $F:M\to N$ has constant rank r, then locally

$$\hat{F}(x^1,\dots,x^r,x^{r+1},\dots,x^m)=(x^1,\dots,x^r,0,\dots,0).$$

In particular:

- F submersion: $\hat{F}(x^1,\dots,x^n,x^{n+1},\dots,x^m)=(x^1,\dots,x^n)$ $\quad(m>n)$,
- F immersion: $\hat{F}(x^1,\dots,x^m)=(x^1,\dots,x^m,0,\dots,0)$ $\quad(m<n)$.

Proof idea:

- Work in local coordinates, center $p=(0,0)$, $F(p)=(0,0)$. Relabel coordinates s.t. $F(x,y)=(Q(x,y),R(x,y))$ $\left((x,y)=(\underbrace{x^1,\dots,x^r}_{x},\underbrace{x^{r+1},\dots,x^m}_{y})\right)$, with $\frac{\partial Q^i}{\partial x^j}$ non-singular ($\operatorname{rank}F=r$).

$\varphi^{-1}(x,y) := (A(x,y), B(x,y)) \Rightarrow (x,y) = \varphi(A(x,y), B(x,y)) = (Q(A(x,y), B(x,y)), B(x,y)) \Rightarrow y = B(x,y)$
$x = Q(A(x,y),y)$

- Def. $\varphi(x,y) = (Q(x,y), y)$, with inverse fct. thm. $F \circ \varphi^{-1}(x,y) = (x, \tilde{R}(x,y))$, with
$\hookrightarrow = R(A(x,y),y)$

$$D(F\circ\varphi^{-1})(x,y) = \begin{pmatrix} \delta^i_j & 0 \\ \frac{\partial \tilde{R}^i}{\partial x^j} & \frac{\partial \tilde{R}^i}{\partial y^j} \end{pmatrix}, \text{ so } \frac{\partial \tilde{R}^i}{\partial y^j} = 0$$ since diffeomorphism φ does not change rank $\Rightarrow \tilde{R}$ indep. of y

- With another diffeomorphism ψ we get $\psi \circ F \circ \varphi^{-1}(x,y) = (x,0)$. (Use $\psi(v,w) = (v, w - S(v))$, $S(x) = \tilde{R}(x,0)$.)