Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 10, Mar. 3, 2023

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3.2 Submanifolds

Recall: M smooth m-manifold with atlas $\mathcal{A}$, $U \subset M$ open.

Def. atlas $\mathcal{A}_U = \{ (V \cap U, \varphi|_{V\cap U}) : (V,\varphi) \in \mathcal{A} \}$.

$\Rightarrow$ U is also a smooth m-manifold, called open submanifold of M.

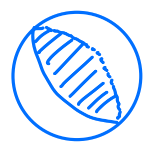
We want to consider more general submanifolds, e.g., torus as submanifold of $\mathbb{R}^3$ (which is not open in $\mathbb{R}^3$)

Note: • We identify $\mathbb{R}^k$ with $\mathbb{R}^n$, $k<n$: $\{ (x^1,\dots,x^k,x^{k+1},\dots,x^n) : x^{k+1} = \dots = x^n = 0 \} \subset \mathbb{R}^n$

• A k-slice of open $U \subset \mathbb{R}^n$ is $\{ (x^1,\dots,x^k,x^{k+1},\dots,x^n) \in U : x^{k+1} = c^{k+1},\dots, x^n = c^n \}$

Def.: Let N^n be a smooth manifold, $M \subset N$. M is called embedded submanifold of dimension $m \leq n$ if $\forall p \in M$ there is a coordinate chart (V,ψ) of N, $p \in V$, $\psi(p) = 0$, s.t.

$$\psi(M \cap V) = \underbrace{\{ (x^1,\dots,x^m,x^{m+1},\dots,x^n) \in \psi(V) : x^{m+1} = \dots = x^n = 0 \}}_{m\text{-slice of } \psi(V) \subset \mathbb{R}^n}$$



Note: • M is indeed a manifold (with the subspace topology) and has a smooth structure

• one can show that inclusion map $i : M \hookrightarrow N$ is an embedding (hence the name)

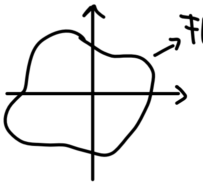
Next: How to characterize embedded submanifolds?

A) Images of certain immersions.

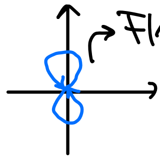
B) Certain level sets $F^{-1}(\{q\}) \subset M$ for $F: M \to N$, $q \in N$.

Recall: $F: M \to N$ smooth

- (smooth) immersion: dF_p injective $\forall p$; embedding: immersion + F homeomorphism



$F(M)$ submanifold



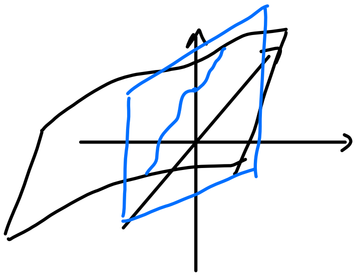
$F(M)$ not submanifold (F immersion, but not embedding, bc. F not homeom.: $F((-\pi,\pi))$ is compact)

(If we remove $(0,0)$ in small neighborhd. $\Rightarrow$ 4 connected comp.'s)

$F: (-\pi,\pi) \to \mathbb{R}^2,\ t \mapsto (\sin 2t, \sin t)$

$\Rightarrow$ $F(M)$ should be a submanifold if F is an embedding

- (smooth) submersion: dF_p surjective $\forall p$



$F: \mathbb{R}^2 \to \mathbb{R}$: no $dF_p = $ ▱ (plane parallel to x-y plane)

$\Rightarrow$ level sets $F^{-1}(\{q\})$ submanifolds

↳ should also be true if F is not a submersion, but dF_p surjective $\forall p \in F^{-1}(\{q\})$

A)

Proposition: If $F: M \to N$ is an embedding, then $F(M)$ is an embedded submanifold of N, and $F: M \to F(M)$ a diffeomorphism.

Proof: Consider $q = F(p)$, centered charts (U,φ) at p, (V,ψ) at q s.t. $F(U) \subset V$.

Rank-thm. for embedding F $\Rightarrow$ $\psi \circ F \circ \varphi^{-1}(x^1,\dots,x^m) = (x^1,\dots,x^m,0,\dots,0)$

Now: $F(U) \subset F(M)$ open (F homeo.) in subspace top. of $F(M)$ $\Rightarrow$ $\exists$ neighborhood $W \ni q$ s.t. $F(U) = F(M) \cap W$.

Take $W \subset V$ $\Rightarrow$ $\psi \circ F(U) = \psi(F(M) \cap W) = m$-slice of $\psi(W)$.

Diffeomorphism: basically clear from def.: $F^{-1}: F(M) \to M$ smooth since F immersion $\square$

Proposition: If $F: M \to N$ is a smooth injective immersion and M compact, then $F(M)$ is an embedded submanifold.

Proof: M compact, N Hausdorff $\Rightarrow F: M \to N$ an open map (maps open sets into open sets) $\square$

($K \subset M$ closed $\overset{M \text{ comp.}}{\Longrightarrow}$ K comp. $\overset{F \text{ cont.}}{\Longrightarrow}$ $F(K)$ comp. $\overset{N \text{ Hausd.}}{\Longrightarrow}$ $F(K)$ closed $\Rightarrow$ F closed map

Since $F: M \to F(M)$ also bijective $\Rightarrow$ $F: M \to F(M)$ homeom. $\Rightarrow$ F embedding)