Topology and Manifolds, Constructor University Bremen, Spring 2023 Session 11
Prof. Dr. Soeren Petrat Mar. 9, 2023

Next: Embedded submanifolds from level sets.

Proposition: If $F: M \to N$ is smooth with constant rank r, $q \in N$, then $F^{-1}(\{q\})$ is an embedded submanifold of M of dimension $\dim M - r$.

Proof: Let $p \in F^{-1}(\{q\})$, choose centered charts (U, φ) and (V, ψ) containing p and q, i.e., in particular, $\varphi(p) = 0$, $\psi(q) = 0$.

Rank Thm. $\Rightarrow$ $\psi \circ F \circ \varphi^{-1}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, \underset{\overset{\uparrow}{n}}{0})$

$\Rightarrow F^{-1}(\{q\}) \cap U = \{(0, \dots, 0, x^{r+1}, \dots, x^m)\} \Rightarrow m-r$ slice $\square$

Note: in particular: F submersion $\Rightarrow F^{-1}(\{q\})$ embedded submanifold.

Next: need only check surjectivity of dF_p for $p \in F^{-1}(\{q\})$.

Def.: Let $F: M \to N$ be smooth.

- If dF_p is surjective for some $p \in M$, p is a **regular point** of F; otherwise p is a **critical point** of F.
- If all $F^{-1}(\{q\})$ are regular points, $q \in N$ is called **regular value**; if not, q is called **critical value**.

Note: $\dim M < \dim N \Rightarrow$ all $p \in M$ are critical points

Proposition: If $F: M \to N$ smooth, $q \in F(M)$ a regular value, then $F^{-1}(\{q\})$ is an embedded submanifold of dimension $\dim M - \dim N$.

Proof: similar to before by Rank Thm.

Ex.: n-sphere: consider $F: \mathbb{R}^{n+1} \to \mathbb{R}$, $F(x^1, \dots, x^{n+1}) = \sum_{j=1}^{n+1} (x^j)^2$

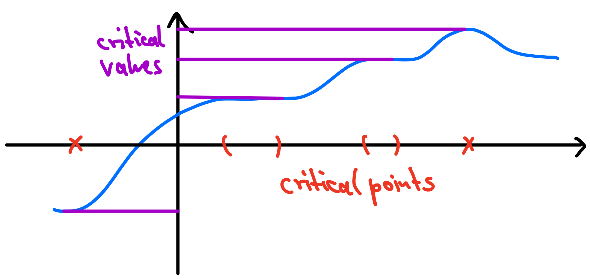
$\Rightarrow dF_x = 2(x^1, \dots, x^{n+1})$

$\Rightarrow \operatorname{rank} dF_x = 1$ for $x \neq 0$ $\Rightarrow$ any $x \neq 0$ is a regular point

$\Rightarrow F^{-1}(\{q\})$ for any $\mathbb{R} \ni q \neq 0$ is an n-dim. embedded submanifold of $\mathbb{R}^{n+1}$

Ex.: For $\Phi(x,y) = x^2 - y^2$, the level set $\Phi^{-1}(\{0\})$ is <u>not</u> an embedded submanifold.

3.3 Sard's Theorem



Sard: "critical values (of smooth fct.s) have measure 0"

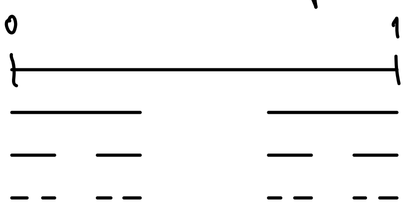
Recall from Analysis II:

- box in $\mathbb{R}^n$: $R = [a_1,b_1] \times \dots \times [a_n,b_n]$
- volume of R = Lebesgue measure $\lambda(R) = (b_1-a_1)\cdot \dots \cdot (b_n-a_n)$

<u>Def.</u>: Any $A \subset \mathbb{R}^n$ has Lebesgue measure zero if for any $\varepsilon > 0$ there exist countable boxes $R_1, R_2, \dots$ such that $A \subset \bigcup_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} \lambda(B_i) < \varepsilon$.

Note: could also take balls instead of boxes

Ex.:
- A countable has measure 0 (choose points as boxes)
- Cantor set:



start with $[0,1]$, always cut out the middle thirds

$$\begin{aligned} \Rightarrow \text{volume} &= 1 - \left(\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \dots\right) \\ &= 1 - \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k \\ &= 1 - \frac{1}{3}\,\frac{1}{1-\frac{2}{3}} \\ &= 0 \end{aligned}$$

but Cantor set is actually uncountable

Lemma: Countable unions of sets of measure zero have measure zero.

Proof: Let $\varepsilon > 0$, call the sets of measure zero A_i.

Choose boxes $R_{i,1}, R_{i,2}, \dots$ to cover A_i s.t. $\sum_{j=1}^{\infty} \lambda(R_{i,j}) < \frac{\varepsilon}{2^i}$

$\Rightarrow \{R_{i,j}\}_{i,j \in \mathbb{N}}$ covers $\bigcup_{i \in \mathbb{N}} A_i$

$\Rightarrow \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \lambda(R_{i,j}) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$ $\square$

Sard's Theorem: Let $U \subset \mathbb{R}^m$ be open and $f: U \to \mathbb{R}^n$ be smooth. Then the set of critical values of f has Lebesgue measure zero.

Note: For $n > m$, this means $f(U)$ has measure zero.

Remark: We only prove the $\mathbb{R}^n$ version. For manifolds, we take countably many charts, then use the lemma above, and use that measure zero is diffeomorphism invariant.

The result is: The set of critical values of $F: M \to N$ has measure zero.