Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 12, Mar. 10, 2023

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Today we prove:

<u>Sard's Theorem:</u> Let $U \subset \mathbb{R}^m$ be open and $f: U \to \mathbb{R}^n$ be smooth. Then the set of critical values of f has Lebesgue measure zero.

<u>Proof for $m<n$ and $m=n$:</u>

<u>$m<n$:</u> Idea: boxes in $\mathbb{R}^m$ are smaller than boxes in $\mathbb{R}^n$ e.g., $\bigcup_{n=3}^{\infty}\left[\frac{1}{n}, 1-\frac{1}{n}\right] = (0,1)$

From topology in $\mathbb{R}^m$: we know that we can write U as countable union of cubes

$$R = [a_1, a_1+\delta] \times \dots \times [a_m, a_m+\delta] \subset U$$

$\Rightarrow$ If $\lambda(f(R)) = 0$, we are done (by previous Lemma)

Mean-value thm.: $\|f(x)-f(y)\| \leq K\|x-y\| \quad \forall x,y \in R$ for some $K>0$

uniform in x,y, since R compact

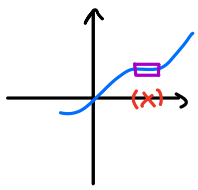
Now choose $N \geq 1$, divide R into smaller cubes R_j with side length $\frac{\delta}{N}$ $\Rightarrow N^m$ cubes

$\Rightarrow f(R_j) \subset$ ball of radius $K\frac{\delta}{N}$, with volume $C\left(\frac{\delta}{N}\right)^n$ for some $C>0$.

$\Rightarrow f(R) \subset \bigcup_j R_j$, with volume $\leq N^m C\left(\frac{\delta}{N}\right)^n = C\delta^n N^{m-n}$ (recall $m<n$).

$\Rightarrow$ This holds for arbitrary (large) N, so $\lambda(f(R))$ can be made arbitrarily small.

$\underline{m=n}$: Idea: image of a ball around critical point contained in small cylinder



Choose cube R as above, $C := \{\text{critical points}\}$ $(f(C) = \text{critical values})$, show $\lambda(f(R \cap C)) = 0$

$\hookrightarrow$ divide R into N^n rectangles (side length $\frac{\delta}{N}$) R_j

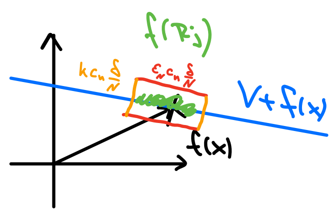
By def. of derivative: $\|f(x) - f(y) - Df(x)(x-y)\| \leq \varepsilon_N \|x-y\|$ $\forall x,y \in R_j$, and $\varepsilon_N \xrightarrow{N\to\infty} 0$
(see also HW1, Problem 3)
$\leq \varepsilon_N C_n \frac{\delta}{N}$ for some C_n (e.g., $C_n = \sqrt{n}$)

Next: fix $x \in R_j \cap C$ (in case this is non-empty)

Then $Df(x)$ not surjective $\Rightarrow \{Df(x)(x-y) : y \in R_j\} \subset V$ = some $n-1$ dim. subspace of $\mathbb{R}^n$

$\Rightarrow \{f(x) - f(y) : y \in R_j\}$ has distance $\varepsilon_N C_n \frac{\delta}{N}$ from V

$\Rightarrow \{f(y) : y \in R_j\}$ has distance $\varepsilon_N C_n \frac{\delta}{N}$ from hyperplane $V + f(x)$



Mean-value thm.: $\|f(x) - f(y)\| \leq K\|x-y\| \leq K C_n \frac{\delta}{N}$

$\Rightarrow \{f(y) : y \in R_j\} \subset$ cylinder of height $2\varepsilon_N C_n \frac{\delta}{N}$ and base = $(n-1)$-sphere of radius r,
with $r \leq K C_n \frac{\delta}{N}$

$\Rightarrow$ volume of cylinder $\leq C\left(\frac{\delta}{N}\right)^{n-1} \frac{\delta}{N} \varepsilon_N$ for some $C > 0$.

$\Rightarrow f(R \cap C) \subset \bigcup$ cylinders, with volume $\leq C\left(\frac{\delta}{N}\right)^n \hat{\varepsilon}_N \underbrace{N^n}_{\#\text{ of cubes } R_j^n} = C\delta^n \varepsilon_N$ $\square$

$\hat{\varepsilon}_N = \max_j \varepsilon_N^{(j)}$ ($\varepsilon_N^{(j)}$ the ε_N for R_j)

Remarks: • C^1 is enough instead of smoothness in our proof of $m \le n$, but for $m > n$, need C^k with $k > \max(0, m-n)$

($\exists$ example of $f: \mathbb{R}^2 \to \mathbb{R}$ C^1 with $f(C) \supset$ interval, due to Whitney)

• Continuity not enough: space-filling curves $F: [0,1] \to [0,1]^2$ surjective