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Prof. Dr. Soeren Petrat

3.4 Whitney Embedding Theorem

Result: embedding of a smooth m-dim. manifold M into $\mathbb{R}^{2m+1}$

Two steps: 1) If $\exists$ embedding into $\mathbb{R}^N$ for $N > 2m+1$, then $\exists$ embedding into $\mathbb{R}^{N-1}$.

2) Embedding into some $\mathbb{R}^N$

Here: consider only M compact, since then in particular:

Recall: M compact, $F: M \to N$ smooth injective $\underbrace{\text{immersion}}_{dF_p \text{ injective } \forall p\in M}$ $\Rightarrow$ F embedding

First, two corollaries of Sard's thm.:

Corollary: Let $F: M \to N$ be smooth. Then the set of regular values of F is dense.
(dense = interior of complement empty).

Proof: Sard: Set of critical values has measure 0. In general, complements of sets of measure 0 are dense. $\square$

This implies:

Lemma: Let M be a compact smooth m-dim. manifold, N smooth n-dim. manifold, $m < n$, and $F: M \to N$ smooth. Then $N \setminus F(M)$ is a dense open subset of N.

Proof: $m<n$, so every $p\in M$ is a critical point $\Rightarrow F(M)$ has measure zero.

By previous corollary regular values $R_F = N\setminus F(M)$ dense.

M compact $\Rightarrow F(M)$ closed $\Rightarrow N\setminus F(M)$ open. $\square$

Next, we prove step 1) by projecting F on a lower dim. space.

Def.: Let $v\in S^{N-1}\subset\mathbb{R}^N$, then we def. the orthogonal projection

$$\pi_v(x) = x - \langle x,v\rangle v \quad \forall x\in\mathbb{R}^N.$$

Note: $\pi_v(v) = v - \underbrace{\langle v,v\rangle}_{=1\ (v\in S^{N-1})} v = 0$

Lemma: Let M be a compact m-dim. smooth manifold, $F: M\to\mathbb{R}^N$ a smooth injective immersion, and $N>2m+1$. Then there is a dense set of vectors $v\in S^{N-1}$ s.t. $\pi_v\circ F$ is a smooth inj. immersion $M\to\mathbb{R}^{N-1}$.

Proof: set $X = F(M)\subset\mathbb{R}^N$

- $\pi_v\circ F$ injective: need $\pi_v(x)\neq\pi_v(y)\ \forall x,y\in X$, i.e.,

$$x-\langle x,v\rangle v \neq y - \langle y,v\rangle v \iff \frac{x-y}{\|x-y\|}\neq v$$

Now def. $\Delta_X = \{(x,x): x\in X\}$, the diagonal of $X\times X$, and def.

$$h: \underbrace{X\times X\setminus\Delta_X}_{\dim = 2m} \to \underbrace{S^{N-1}}_{\dim = N-1},\quad h(x,y) = \frac{x-y}{\|x-y\|}$$

Need $v\notin h(X\times X\setminus\Delta_X)$

By previous lemma a dense set of such v exists as long as $2m<N-1$.

- $\pi_v\circ F$ immersion: HW

Theorem (Whitney embedding, compact case):

Any compact smooth m-dim. manifold M can be embedded into $\mathbb{R}^{2m+1}$.

Proof: Show embedding into some $\mathbb{R}^d$, then $d = 2m+1$ follows from previous lemma.

M compact $\Rightarrow$ can choose finite open cover $\{U_i\}_{i=1,\dots,N}$, corresponding charts (U_i, φ_i).

Choose new open cover $\{V_i\}$ s.t. $\overline{V_i} \subset U_i$.

Def. bump fct.s $\rho_i : M \to \mathbb{R}$, s.t. $\rho_i|_{\overline{V_i}} = 1$ and $\operatorname{supp} \rho_i \subset U_i$

Def. $F := (\rho_1 \varphi_1, \dots, \rho_N \varphi_N, \rho_1, \dots, \rho_N)$ $\left(\rho_i \varphi_i|_{(\operatorname{supp} \rho_i)^c} := 0\right)$ $(F : M \to \mathbb{R}^{mN+N})$

HW: show that this F is injective and an immersion.

Since M compact, F is an embedding. $\square$