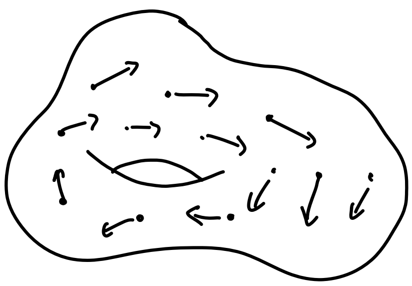
Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 15

Prof. Dr. Soeren Petrat — Mar. 23, 2023

5. Vector Fields and Differential Forms

5.1 Vector Fields



Attach a vector to each $p \in M$.

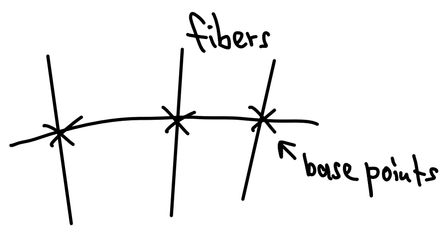
Vectors at $p \in M$ should live in T_pM.

Def.: Let M be a smooth manifold, then the $\boxed{\text{tangent bundle}}$ is def. as

$$TM := \underbrace{\bigsqcup_{p\in M} T_pM}_{= \dot{\bigcup}_{p\in M} T_pM \text{ (disjoint union)}} = \{(p,v) : p \in M, v \in T_pM\}$$

Note:

- A natural projection map is $\pi : TM \to M$, $\pi(p,v) = p$.
  We call p "base point" and $\pi^{-1}(p)$ "fiber".



- Simple ex.: $T\mathbb{R}^n = \bigsqcup_{a\in\mathbb{R}^n} T_a\mathbb{R}^n \cong \bigcup_{a\in\mathbb{R}^n} \{a\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, but usually TM is not a cartesian product (no canonical identification of different T_pM's).

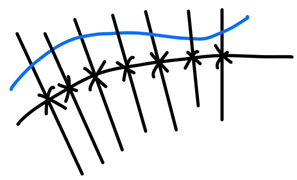
- We will sometimes identify $(p,v) \in TM$ with $v \in T_pM$ if no confusion arises.

Proposition: Let M be a smooth n-manifold, then TM has a natural topology and smooth structure that make it into a $2n$-dim. smooth manifold. Furthermore, the projection π is smooth.

Proof idea:
- take chart (U,φ) with $p\in U$
- def. $\tilde{\varphi}: \underset{\subset TM}{\pi^{-1}(U)} \to \mathbb{R}^{2n}$, $\tilde{\varphi}(p,v) = \big(\varphi(p), \overbrace{d\varphi_p(v)}^{T_pM \to T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n}\big) \in \underset{\subset \mathbb{R}^{n}}{\varphi(U)} \times \mathbb{R}^n$
- smoothness of transition maps can be checked $\square$

Def.: A smooth vector field is a smooth map $X: M \to TM$, s.t. $X(p) \in T_pM$ $\forall p\in M$ (i.e., $\pi \circ X = id_M$).

Note: • X is also called section of the map $\pi: TM \to M$



Note: • zero section $X(p) = 0_{T_pM}$, not a constant map (0_{T_pM} can vary for different p)

- open $U \subset \mathbb{R}^n$, $T_pU \cong \mathbb{R}^n$, def. vector fields $\frac{\partial}{\partial x^i}(p) = e_i$ (e_i = i-th canonical basis vector of $\mathbb{R}^n$)

$\Rightarrow \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p$ basis of T_pU $\forall p \in U$

$\Rightarrow$ Any vector field can be written in local coordinates as $X(p) = \sum_{i=1}^{n} f^i(p) \frac{\partial}{\partial x^i}(p)$,

f^i = smooth component fct.s