Topology and Manifolds, Constructor University Bremen, Spring 2023 Session 16
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last time: • tangent bundle $TM = \{(p,v) : p \in M, v \in T_pM\}$
↳ a smooth $2n$-manifold (M smooth n-manifold)

• vector field: $X: M \to TM$ s.t. $\pi \circ X = \mathrm{id}$ (i.e., $X(p) \in T_pM \ \forall p \in M$)

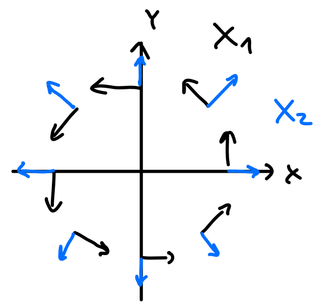
Choosing chart (U, φ), we can write locally (= in this coordinate chart): $X(p) = \sum_{i=1}^n X^i(p) \frac{\partial}{\partial x^i}\Big|_p$
smooth component fct.s

Examples of vector fields:

• $M \subset \mathbb{R}^n$ open, $v \in M$, then $X: M \to TM$, $X(p) = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p = \langle v, \nabla_p \rangle$ is a vector field, called gradient vector field

• $M = \mathbb{R}^2 \setminus \{0\}$, $X_1 = -\frac{y}{r}\frac{\partial}{\partial x} + \frac{x}{r}\frac{\partial}{\partial y}$, $r = \sqrt{x^2+y^2}$

$$X_2 = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y}$$



$\Rightarrow$ called orthonormal frame, since $X_1(p)$ and $X_2(p)$ orthonormal $\forall p \in M$ (as vectors in $\mathbb{R}^2$)

• $F: M \to N$ smooth, $X: M \to TM$ vector field

$\Rightarrow dF_p : T_pM \to T_{F(p)}N$, so def. $dF_p(X(p)) \in T_{F(p)}N \longrightarrow$ not necessarily a vector field on N
(e.g., if F not injective or not surjective)

But if F is a diffeomorphism, we have that the **push-forward**

$F_* X : N \to TN$, $F_*X(q) = dF_{F^{-1}(q)}(X(F^{-1}(q)))$ is a vector field

(F_*X is smooth since $F_*X : N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN$, i.e., composition of smooth maps) frakturX, gothicX

<u>Def.:</u> $\mathfrak{X}(M) := \{\text{all vector fields on } M\}$

Note: • $\mathfrak{X}(M)$ is a vector space: $(aX+bY)(p) = aX(p) + bY(p)$

• $f \in C^\infty(M)$ $(f: M \to \mathbb{R})$, $X \in \mathfrak{X}(M)$ $\Rightarrow$ $fX: M \to TM$, $(fX)(p) = f(p)X(p)$

also a vector field

Next: $\mathfrak{X}(M) \overset{?}{\longleftrightarrow} C^\infty(M)$

$X \in \mathfrak{X}(M)$, $f \in C^\infty(U)$, $U \subset M$, then $Xf: U \to \mathbb{R}$, $\underbrace{(Xf)}_{\text{not multiplication!}}(p) := \overbrace{X(p)}^{\in T_pM} f$ is again a smooth function.

In local coordinates: $(Xf)(p) = \sum_{i=1}^{n} X^i(p) \frac{\partial f}{\partial x^i}\Big|_p$, so Xf is derivative of f in direction $X(p)$

$\Rightarrow L_X: C^\infty(M) \to C^\infty(M)$, $L_X f = Xf$

↳ linear

↳ $L_X(fg)(p) = X(fg)(p) = X(p)(fg) \overset{X(p) \text{ derivation at } p}{=} f(p)\big(X(p)g\big) + g(p)\big(X(p)f\big)$

$\Rightarrow L_X(fg) = f L_X g + g L_X f$

<u>Def.:</u> If $D: C^\infty(M) \to C^\infty(M)$ is linear and satisfies product rule, D is called **(global) derivation.**

<u>Proposition:</u> $D: C^\infty(M) \to C^\infty(M)$ derivation $\Leftrightarrow$ $Df = Xf$ for some $X \in \mathfrak{X}(M)$

Proof: "$\Leftarrow$" done, for "$\Rightarrow$" def. $X(p)(f) := (Df)(p)$

↳ $X(p): C^\infty(M) \to \mathbb{R}$ indeed a derivation $(X(p) \in T_pM)$

↳ smoothness can be checked (X smooth $\Leftrightarrow$ Xf smooth $\forall f$) $\square$

<u>Proposition:</u> $X, Y \in \mathfrak{X}(M) \Rightarrow f \mapsto X\underbrace{Yf}_{\in C^\infty(M)} - YXf$ is a global derivation

Proof: linearity clear.

Product rule: $XY(fg) - YX(fg) = X(fYg + gYf) - Y(fXg + gXf)$

$$= Xf\,Yg + fXYg + Xg\,Yf + gXYf$$
$$- Yf\,Xg - fYXg - Yg\,Xf - gYXf$$
$$= fXYg + gXYf - fYXg - gYXf$$
$$= f(XY - YX)g + g(XY - YX)f$$

$\square$

<u>Def.:</u> $X, Y \in \mathfrak{X}(M)$, then $[X,Y]: C^\infty(M) \to C^\infty(M)$, $[X,Y]f = XYf - YXf$ is called

$\boxed{\text{Lie bracket.}}$

Note: · $[X,Y]$ is a vector field

· $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$, $Y = \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j} \Rightarrow [X,Y] = \sum_{i,j=1}^n \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$ (check!)

<u>Proposition:</u> For $X, Y, Z \in \mathfrak{X}(M)$ we have

a) bilinearity: $[aX + bY, Z] = a[X,Z] + b[Y,Z]$

$[Z, aX + bY] = a[Z,X] + b[Z,Y]$ $\quad \forall a,b \in \mathbb{R}$

b) antisymmetry: $[X,Y] = -[Y,X]$

c) Jacobi identity: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

d) for all $f, g \in C^\infty(M)$: $[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X$

e) for all diffeomorphisms $F: M \to N$: $F_*[X,Y] = [F_*X, F_*Y]$

Proof: HW

<u>Def.:</u> A Lie algebra (over $\mathbb{R}$) L is a vector space with a bracket $[\cdot,\cdot] : L \times L \to L$ that satisfies a), b), c) from above.

Ex.:
- $\mathfrak{X}(M)$
- $M_{n\times n}(\mathbb{R})$ with commutator $[A, B] = AB - BA$
- Any vector space V with $[x,y] := 0$ is a Lie algebra