Topology and Manifolds, Constructor University Bremen, Spring 2023 — Session 17, Mar. 30, 2023

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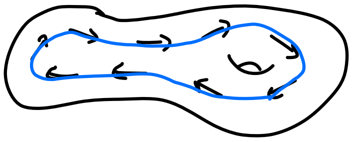
5.2 Integral Curves



M smooth manifold, $I \subset \mathbb{R}$ open interval, curve $\gamma: I \to M$

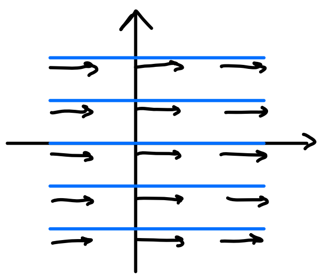
$\Rightarrow$ Velocity at t_0 is $\dot\gamma(t_0) = \gamma'(t_0) := d\gamma\left(\frac{d}{dt}\Big|_{t_0}\right) \in T_{\gamma(t_0)}M$

i.e., $\gamma'(t_0) f = d\gamma\left(\frac{d}{dt}\Big|_{t_0}\right) f = \frac{d}{dt}\Big|_{t_0} (f \circ \gamma) = \underbrace{(f\circ\gamma)}_{\text{derivative of } f\circ\gamma : I \to \mathbb{R}}{}'(t_0)$

In local coordinates: $\gamma'(t_0) = \sum_i \frac{d\gamma^i}{dt}(t_0) \frac{\partial}{\partial x^i}\Big|_{\gamma(t_0)}$

Def.: A smooth curve $\gamma: I \to M$ is called **integral curve** of the vector field $X \in \mathfrak{X}(M)$ if $\gamma'(t) = X(\gamma(t))$ $\forall t \in I$.

Ex.: • $M = \mathbb{R}^2$, $X = \frac{\partial}{\partial x^1}$ $\Rightarrow$ $X(\gamma(t)) = \frac{\partial}{\partial x^1}\Big|_{\gamma(t)}$



$$\gamma'(t) = \underbrace{\frac{d\gamma^1}{dt}}_{\overset{!}{=}1} \frac{\partial}{\partial x^1}\Big|_{\gamma(t)} + \underbrace{\frac{d\gamma^2}{dt}}_{\overset{!}{=}0} \frac{\partial}{\partial x^2}\Big|_{\gamma(t)}$$

$\Rightarrow$ integral curves are $\gamma(t) = \begin{pmatrix} a+t \\ b \end{pmatrix}$ for $a, b \in \mathbb{R}$

• $M = \mathbb{R}^2$, $X = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}$

$$\Rightarrow \dot\gamma^1(t) \frac{\partial}{\partial x^1}\Big|_{\gamma(t)} + \dot\gamma^2(t) \frac{\partial}{\partial x^2}\Big|_{\gamma(t)} = \gamma^1(t) \frac{\partial}{\partial x^2}\Big|_{\gamma(t)} - \gamma^2(t) \frac{\partial}{\partial x^1}$$

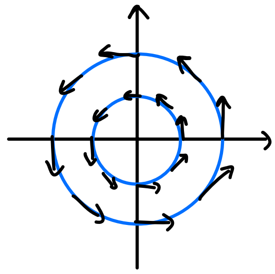
$\Rightarrow$ need to solve system of two ODEs:

$\dot\gamma^1(t) = -\gamma^2(t)$ $\quad (\Rightarrow \ddot\gamma^1 = -\dot\gamma^2 = -\gamma^1)$

$\dot\gamma^2(t) = \gamma^1(t)$ $\quad (\Rightarrow \ddot\gamma^2 = \dot\gamma^1 = -\gamma^2)$

$\Rightarrow$ solution: $\gamma(t) = \begin{pmatrix} a\cos t - b\sin t \\ a\sin t + b\cos t \end{pmatrix}$, circles

$= \underbrace{\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}}_{\text{rotation}} \begin{pmatrix} a \\ b \end{pmatrix}$ (and $\gamma(t) = (0,0)$)



$\Rightarrow$ Finding integral curves = solving system of ODEs in local coordinates:

$$\dot\gamma^i(t) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)} = X^i(\gamma(t)) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

$$\Rightarrow \quad \dot\gamma^i(t) = X^i(\gamma^1(t), \dots, \gamma^n(t)), \quad i = 1, \dots, n \quad \text{(autonomous ODEs)}$$

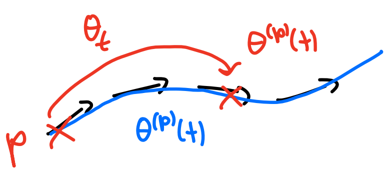
<u>Proposition:</u> Let $X \in \mathfrak{X}(M)$ for a smooth manifold M. Then $\forall p \in M$ there is $\varepsilon > 0$ and a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ that is an integral curve of X with $\gamma(0) = p$.

Proof notes:

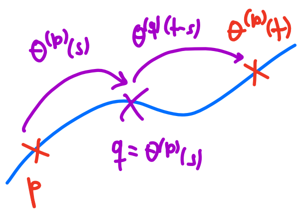
- This is a classical result for $M = \mathbb{R}^n$, which is proved using the Banach fixed point theorem.
- Since this is a local result only, this proves the proposition for any M by choosing local coordinates.

Next: consider vector field X and an integral curve $\theta^{(p)}(t)$ starting at p ($\theta^{(p)}(0) = p$)

Now fix t, def. $\theta_t : M \to M$, $\theta_t(p) = \theta^{(p)}(t)$ (assume this exists)



Note:



$\Rightarrow \quad \theta_s \theta_{t-s} = \theta_t$, or $\theta_t \circ \theta_s = \theta_{t+s}$

Def.: $\Theta: \mathbb{R} \times M \to M$ smooth is called $\boxed{\text{global flow}}$ if $(\Theta(t,p) =: \Theta_t(p))$

or "one-parameter group action"

- $\Theta_0(p) = p \quad \forall p \in M$
- $\Theta_t(\Theta_s(p)) = \Theta_{t+s}(p) \quad \forall p \in M,\ t,s \in \mathbb{R}$

The map $X: M \to TM$, $X(p) := \underbrace{\Theta^{(p)\prime}(0)}_{\text{velocity at } t=0 \text{ of curve with starting point } p}$ is called $\boxed{\text{infinitesimal generator}}$ of Θ.

One can show that this X is indeed a smooth vector field, and $\Theta^{(p)}$ are its integral curves.

Ex.: • $M = \mathbb{R}^2$, $V = \frac{\partial}{\partial x} \Rightarrow$ flow $\tau: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $\tau_t(x,y) = \begin{pmatrix} x+t \\ y \end{pmatrix}$

But if $M = \mathbb{R}^2 \setminus \{0\}$ flow is not global.

$\Big[\ w = x^2 \frac{\partial}{\partial x} \Rightarrow \frac{dx}{dt} = x^2 \Rightarrow \gamma(t) = \left(\frac{1}{1-t}, 0\right)$, $\gamma(0) = (1,0)$

Def.: $\Theta: (-\varepsilon,\varepsilon) \times U \to M$ smooth is called $\boxed{\text{local flow}}$ if
$U \subset M$ open

- $\Theta_0(p) = p \quad \forall p \in U$
- $\Theta_t(\Theta_s(p)) = \Theta_{t+s}(p)$.
  whenever this exists

Fundamental Theorem on Flows:

For any $X \in \mathfrak{X}(M)$ $\exists$ unique local flow Θ s.t. $\Theta(p)$ are the integral curves of X starting at $p \in M$. X is the infinitesimal generator of Θ.

Def.: $X \in \mathfrak{X}(M)$ is called $\boxed{\text{complete}}$ if it generates a global flow.

Proposition: For compact smooth manifolds M, any vector field is complete.

Proof sketch: compactness $\Rightarrow$ finite cover $\Rightarrow$ patch together local domains of flows.

Next: Derivative of a vector field X in direction of another vector field Y.

Note: Directional derivatives of vector field in $\mathbb{R}^n$:

$$\lim_{t\to 0} \frac{X(p+tv)-X(p)}{t} = \frac{d}{dt} X(p+tv)\Big|_{t=0} = \begin{pmatrix} \vdots \\ \sum_{j=1}^n \frac{\partial X^i}{\partial x^j} v^j \\ \vdots \end{pmatrix}$$

What about directional derivatives of vector fields in M?

Replace $X(p+tv)$ by $X(\theta_t(p))$, for some flow θ of vector field Y.

But still $X(p) \in T_pM$, $X(\theta_t(p)) \in T_{\theta_t(p)}M$, so how to identify tangent spaces?

$$\underset{\uparrow \text{ fixed } t}{\theta_t} : M \to M, \text{ so } d(\theta_t)_p : T_pM \to T_{\theta_t(p)}M, \text{ and } \left(d(\theta_t)_q\right)^{-1} = d(\theta_t^{-1})_q = d(\theta_{-t})_q$$

maps $T_{\underbrace{\theta_t(p)}_{q}} M \to T_pM$

$\Rightarrow$ We def. the Lie derivative of X with respect to Y as

$$\mathcal{L}_Y X(p) = \lim_{t\to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\left(X(\theta_t(p))\right) - X(p)}{t} = \frac{d}{dt}\Big|_{t=0} d(\theta_{-t})_{\theta_t(p)}\left(X(\theta_p(t))\right),$$

$$= \frac{d}{dt}\Big|_{t=0} X(\theta_p(t))(f \circ \theta_{-t})$$

where θ is the flow of Y.

Thm.: $\mathcal{L}_Y X = [Y, X]$

$\Rightarrow$ Lie derivative can be computed without explicitly computing the flow.

Proof: Can be reduced to a direct computation in convenient coordinates.