5.3 Covectors

Let V be a (finite dim. real) vector space

Def.: Any linear map $\omega: V \to \mathbb{R}$ (i.e., real-valued linear functional) is called **covector**.
$V^* =$ dual space of $V = \{\text{all covectors}\}$

Ex.: $V = \{\text{column vectors in } \mathbb{R}^n\}$, then $V^* = \{\text{row vectors in } \mathbb{R}^n\}$

e.g. $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\omega = (\omega^1, \dots, \omega^n) \Rightarrow \omega(v) = (\omega^1, \dots, \omega^n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^n \omega^i v_i = \omega^i v_i$

Einstein summation convention:
Summation implied if same index appears twice, as an upper and lower index

If $\underbrace{e_1, \dots, e_n}_{e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i}$ basis of $V \Rightarrow \underbrace{\varepsilon^1, \dots, \varepsilon^n}_{\varepsilon^i = (0, \dots, 1, \dots, 0)}$ basis of V^*, i.e., $\varepsilon^i(e_j) = \delta^i_j$ $\quad (\varepsilon^i(v) = v^i)$

In general:

Def.: $\varepsilon^1, \dots, \varepsilon^n \in V^*$ called **dual basis** to basis $E_1, \dots, E_n$ of V if $\varepsilon^i(E_j) = \delta^i_j$

Prop.: Dual basis is indeed a basis of V^*. (Proof: easy, linear algebra)

$\Rightarrow$ For $V^* \ni \omega = \omega_i \varepsilon^i$, $V \ni v = v^j E_j$ we have $\omega(v) = \omega_i v^j \varepsilon^i(E_j) = \omega_i v^i$

Def.: Let V, W be vector spaces, $A: V \to W$ linear, then **dual map** $A^*: W^* \to V^*$ is def. by
$(A^* \omega)(v) = \omega(Av)$ for all $\omega \in W^*$, $v \in V$.

Note:
- $(A \circ B)^* = B^* \circ A^*$
- $\exists$ canonical isomorphism $V \to V^{**}$ (canonical = independent of arbitrary choices, e.g., of basis)
- $\exists$ isomorphism $V \to V^*$, but it is not canonical (since it is basis dependent)

Def.: The **cotangent space** at $p \in M$ is $T_p^*M := (T_pM)^*$, $\omega \in T_p^*M$ is called **(tangent) covector at p.**

Note: In coordinates, $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}$ basis of $T_pM \Rightarrow$ dual basis $\{\lambda^i|_p\}$ of T_p^*M

$$\Rightarrow T_p^*M \ni \omega = \omega_i \lambda^i|_p \quad \text{with } \omega_i = \omega\left(\frac{\partial}{\partial x^i}\Big|_p\right)$$

Def.: • $T^*M = \bigsqcup_{p \in M} T_p^*M$ is called **cotangent bundle** of M.

• $\omega : M \to T^*M$ with $\omega(p) \in T_p^*M$ is called **covector field** (= **differential 1-form**)

Note: one can show that T^*M is a smooth manifold (of dim. $2m$).

Note: ω covector field, X vector field

$$\Rightarrow \omega(X) : M \to \mathbb{R}, \quad \omega(X)(p) = \omega(p) X(p)$$

In local coordinates: $\omega(X) = \omega_i X^i$

Now: for $f \in C^\infty(M)$, the differential at p was def. as $df_p : T_pM \to T_{f(p)}\mathbb{R}$

We can also regard the **differential of f** as the covector field df, def. by $df_p(v) = vf$ $\forall v \in T_pM$

i.e., $T_p^*M \ni df_p : T_pM \to \mathbb{R}$ $\Rightarrow$ same map with the identification of $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$

In coordinates: $T_p^*M \ni df_p = A_i(p)\lambda^i|_p$ for some smooth A_i

$$\Rightarrow A_i(p) = df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) := \frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial f}{\partial x^i}(p)$$

$$\Rightarrow df_p = \frac{\partial f}{\partial x^i}(p)\lambda^i|_p$$

for $f = x^j$ we find $dx^j|_p = \lambda^j|_p \Rightarrow \boxed{\lambda^j = dx^j} \Rightarrow \boxed{df = \frac{\partial f}{\partial x^i} dx^i}$

Now, consider smooth $F: M \to N$

Recall that for diffeomorphisms F and $X \in \mathfrak{X}(M)$ we defined the pushforward

$$F_* X : N \to TN, \quad F_* X(q) = dF_{F^{-1}(q)}(X(F^{-1}(q))) \qquad (F_* X \text{ is a new vector field})$$

Now take any smooth $F: M \to N$ (not necessarily a diffeomorphism)

<u>Def.:</u> $F: M \to N$ smooth, $\omega: N \to T^*N$ a covector field, then the $\boxed{\text{pullback of } \omega \text{ by } F}$ is def. as

$$F^*\omega : M \to T^*M, \quad (F^*\omega)_p = dF_p^*(\omega_{F(p)}).$$

Note: one can prove that $F^*\omega$ is a smooth covector field

5.4 Tensors

<u>Def.:</u> • $A: \underbrace{V_1 \times \ldots \times V_k}_{\text{vector spaces}} \to W$ is called $\boxed{\text{multilinear}}$ if

$$A(v_1, \ldots, \lambda_i v_i + \tilde{v}_i, \ldots, v_k) = \lambda_i A(v_1, \ldots, v_k) + A(v_1, \ldots, \tilde{v}_i, \ldots, v_k)$$

• $L(V_1, \ldots, V_k; W) =$ set of all multilinear maps $V_1 \times \ldots \times V_k \to W$

• If $A: V_1 \times \ldots \times V_k \to \mathbb{R}$ and $B: W_1 \times \ldots \times W_l \to \mathbb{R}$ are multilinear then

$A \otimes B : V_1 \times \ldots \times V_k \times W_1 \times \ldots \times W_l \to \mathbb{R}$, $A \otimes B(v_1, \ldots, v_k, w_1, \ldots, w_l) = A(v_1, \ldots, v_k) B(w_1, \ldots, w_l)$

is called $\boxed{\text{tensor product}}$ of A and B.

Ex.: $\omega^j \in V_j^*$, then $\omega^1 \otimes \ldots \otimes \omega^n : V_1 \times \ldots \times V_k \to \mathbb{R}$, $(v_1, \ldots, v_k) \longmapsto \omega^1(v_1) \ldots \omega^k(v_k)$

Note: • $\mathcal{B} = \{ \varepsilon_{(1)}^{i_1} \otimes \ldots \otimes \varepsilon_{(k)}^{i_k} : 1 \le i_1 \le n, \ldots, 1 \le i_k \le n \}$ basis of $L(V_1, \ldots, V_k; \mathbb{R})$

• more abstractly one can define a space $V_1 \otimes \ldots \otimes V_k$; here, just take it as the vector space with basis $\mathcal{C} = \{ E_{(1)}^{i_1} \otimes \ldots \otimes E_{(k)}^{i_k} : 1 \le i_1 \le n, \ldots, 1 \le i_k \le n \}$

space of all formal lin. combinations $v_1 \otimes \ldots \otimes v_k$

$$\Rightarrow \quad L(V,\dots,V;\mathbb{R}) \cong V^* \otimes \dots \otimes V^*$$

<u>Def.:</u> $T^k(V^*) = \underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ times}}$ is called space of covariant k-tensors (or covariant rank-k tensors)

$\Lambda^k(V^*) =$ alternating (antisymmetric) covariant k-tensors, i.e., for $\alpha \in \Lambda^k(V^*)$ we have

$$\alpha(\dots, v_i, \dots, v_j, \dots) = -\alpha(\dots, v_j, \dots, v_i, \dots)$$