5.4 Tensors

Recall: ($V_1, \dots, V_k, W, W_1, \dots, W_\ell$ vector spaces)

- $A: V_1 \times \dots \times V_k \to W$ is called multilinear if
$$A(v_1, \dots, \lambda_i v_i + \tilde{v}_i, \dots, v_k) = \lambda_i A(v_1, \dots, v_k) + A(v_1, \dots, \tilde{v}_i, \dots, v_k)$$
- $L(V_1, \dots, V_k; W)$ = set of all multilinear maps $V_1 \times \dots \times V_k \to W$
- If $A: V_1 \times \dots \times V_k \to \mathbb{R}$ and $B: W_1 \times \dots \times W_\ell \to \mathbb{R}$ are multilinear then
$A \otimes B: V_1 \times \dots \times V_k \times W_1 \times \dots \times W_\ell \to \mathbb{R}$, $A \otimes B(v_1, \dots, v_k, w_1, \dots, w_\ell) := A(v_1, \dots, v_k) B(w_1, \dots, w_\ell)$
is called tensor product of A and B.
- $\mathcal{B} = \{\varepsilon^{i_1}_{(1)} \otimes \dots \otimes \varepsilon^{i_k}_{(k)} : 1 \le i_1 \le n, \dots, 1 \le i_k \le n\}$ is a basis of $L(V_1, \dots, V_k; \mathbb{R})$, and we
some (dual) basis of V^*
thus naturally define the n-fold tensor product of V^*. We have
$L(V, \dots, V; \mathbb{R}) \cong V^* \otimes \dots \otimes V^*$

- $T^k(V^*) = \underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ times}}$ = space of covariant k-tensors
- $\Lambda^k(V^*)$ = alternating covariant k-tensors, i.e., $\alpha \in \Lambda^k(V^*)$ satisfies
$\alpha(\dots, v_i, \dots, v_j, \dots) = -\alpha(\dots, v_j, \dots, v_i, \dots)$

Note: $\Sigma^k(V^*)$ = symmetric covariant k-tensors ($\beta(\dots, v_i, \dots, v_j, \dots) = \beta(\dots, v_j, \dots, v_i, \dots)$)

Ex.:
- For a (real) inner product space V, the inner product is a covariant 2-tensor.
- On $\mathbb{R}^n$, the determinant as a fct. of its column (or row) vectors is an alternating covariant 2-tensor.
- $T^2(V^*)$ = space of bilinear forms on V = $\{\alpha = \alpha_{ij}\, \varepsilon^i \otimes \varepsilon^j\}$

We often need to antisymmetrize covariant tensors.

Def.: The $\boxed{\text{alternation } \mathrm{Alt}: T^k(V^*) \to \Lambda^k(V^*)}$ is def. as

$$(\mathrm{Alt}\,\alpha)(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k} (\operatorname{sgn}\sigma)\,\alpha(v_{\sigma(1)},\dots,v_{\sigma(k)}), \text{ where}$$

- S_k is the symmetric group on k elements, i.e., the set of all permutations of $1,\dots,k$
- $\operatorname{sgn}\sigma = (-1)^\sigma = \begin{cases} 1 & \text{for } \sigma \text{ even} \\ -1 & \text{for } \sigma \text{ odd}\end{cases}$ = sign of the permutation σ

Ex.: $\beta \in T^2(V^*) \Rightarrow (\mathrm{Alt}\,\beta)(v_1,v_2) = \frac{1}{2}\left(\beta(v_1,v_2) - \beta(v_2,v_1)\right)$

Note: Analogously we can define $(\mathrm{Sym}\,\alpha)(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k}\alpha(v_{\sigma(1)},\dots,v_{\sigma(k)})$.

The tensor product of two alternating tensors will in general not be alternating, so we def.:

Def.: For $\omega\in\Lambda^k(V^*)$, $\eta\in\Lambda^\ell(V^*)$, the $\boxed{\text{wedge product}}$ (or exterior product) is def. as

$$\Lambda^{k+\ell}(V^*) \ni \boxed{\omega\wedge\eta = \frac{(k+\ell)!}{\underbrace{k!\,\ell!}} \mathrm{Alt}(\omega\otimes\eta)}$$

← combinatorial factor included for convenience

Note: Analogously, the symmetric product of two symmetric tensors α, β is def. as $\alpha\beta := \mathrm{Sym}(\alpha\otimes\beta)$

Properties of wedge product (straightforward to check, some of it in HW):

- bilinear, associative
- $\omega\wedge\eta = (-1)^{k\ell}\,\eta\wedge\omega$
- $\{\varepsilon^{i_1}\wedge\dots\wedge\varepsilon^{i_k} : i_1<\dots<i_k\}$ is a basis of $\Lambda^k(V^*) \Rightarrow \dim\Lambda^k(V^*) = \binom{n}{k}$ $(n=\dim V)$
- $\omega^1\wedge\dots\wedge\omega^k(v_1,\dots,v_k) = \det(\omega^j(v_i))$

↑ No combinatorial factor here with our convention

$\left(\text{E.g., } \omega^1\wedge\omega^2(v_1,v_2) = 2!\,\frac{1}{2!}\left(\omega^1(v_1)\omega^2(v_2) - \omega^1(v_2)\omega^2(v_1)\right) = \det\omega^j(v_i)\right)$

Next: from k-tensors to k-1 tensors

Def.: For $v \in V$, we define the interior multiplication

$$i_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*),\ \omega \mapsto i_v\omega = v \lrcorner \omega \quad (v \text{ into } \omega) \text{ by}$$

$$\underbrace{(i_v\omega)}_{\in \Lambda^{k-1}(V^*)}(v_1, \dots, v_{k-1}) = \omega(v, v_1, v_2, \dots, v_{k-1})$$

Properties: • $i_v \circ i_v = 0$ (since ω's above are alternating)

• $\omega \in \Lambda^k(V^*)$, $\eta \in \Lambda^\ell(V^*) \implies i_v(\omega \wedge \eta) = (i_v\omega) \wedge \eta + (-1)^k \omega \wedge (i_v\eta)$

(direct computation, see HW)

5.5 Differential Forms

Def.: $\Lambda^k(T^*M) = \bigsqcup_{p \in M} \Lambda^k(T_p^*M) =$ alternating covariant k-tensor bundle on M

A differential k-form is an alternating covariant (smooth) tensor field $M \to \Lambda^k(T^*M)$. (I.e., for ω a differential k-form, we have $\omega_p \in \Lambda^k(T_p^*M)$.)

We denote $\Omega^k(M) := \{\text{all k-forms}\}$.

Remarks: • $\omega \in \Omega^k(M)$, $\eta \in \Omega^k(M)$, then $\omega \wedge \eta$ def. pointwise, i.e., $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$

• In coordinates: $\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \equiv \sum_I{}' \omega_I dx^I$, $I = \{i_1, \dots, i_k\}$ a multi-index

smooth functions: $\omega_{i_1 \dots i_k} = \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right)$

Ex.s on $\mathbb{R}^3$: $\omega = \sin(xy)\, dy \wedge dz$, $\eta = dx \wedge dy \wedge dz$