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Last time: • $\Omega^k(M)$ = space of differential k-forms $\omega: M \to \Lambda^k(T^*M)$ smooth s.t. $\omega_p \in \Lambda^k(T_p^*M)$

• In coordinates: $\omega = \sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k} dx^{i_1}\wedge\dots\wedge dx^{i_k} \equiv \sum_I{}' \omega_I dx^I$

__Def.:__ $F: M\to N$ smooth, $\omega\in\Omega^k(N)$, then the pullback $F^*\omega \in \Omega^k(M)$ is def. as

$$\underbrace{(F^*\omega)_p}_{\in\Lambda^k(T_p^*M)}(v_1,\dots,\underbrace{v_k}_{\in T_pM}) = \underbrace{\omega_{F(p)}}_{\in\Lambda^k(T_{F(p)}^*N)}(dF_p(v_1),\dots,\underbrace{dF_p(v_k)}_{T_{F(p)}N})$$

Rules for computation:

• $F^*(\omega\wedge\eta) = (F^*\omega)\wedge(F^*\eta)$

• In coordinates: $F^*\left(\sum_I{}' \omega_I dy^{i_1}\wedge\dots\wedge dy^{i_k}\right) = \sum_I{}' (\omega_I\circ F)\, d(y^{i_1}\circ F)\wedge\dots\wedge d(y^{i_k}\circ F)$

• M,N smooth n-manifolds, (x^i) some local coordinates on M, (y^i) on N, $u: V\to\mathbb{R}$, $V \subset N$, then

$$F^*(u\, dy^1\wedge\dots\wedge dy^n) = (u\circ F)\det \underbrace{\mathrm{Jac}(F)}_{\text{Jacobian matrix of } F} dx^1\wedge\dots\wedge dx^n$$

Ex.: • HW

• $F: \mathbb{R}^2\to\mathbb{R}^3$, $(u,v)\mapsto F(u,v) = \begin{pmatrix} u\\ v\\ u^2-v^2\end{pmatrix}$, $\omega = y\,dx\wedge dz + x\,dy\wedge dz$ (2-form on $\mathbb{R}^3$).

$$\begin{aligned}
\Rightarrow F^*\omega &= F^*(y\,dx\wedge dz + x\,dy\wedge dz)\\
&= v\,du\wedge d(u^2-v^2) + u\,dv\wedge d(u^2-v^2)\\
&= v\,du\wedge(2u\,du - 2v\,dv) + u\,dv\wedge(2u\,du-2v\,dv)\\
&= -2v^2\,du\wedge dv + 2u^2\,dv\wedge du\\
&= -2(u^2+v^2)\,du\wedge dv \quad \text{(2-form on } \mathbb{R}^2)
\end{aligned}$$

Recall: $f \in C^\infty(M) = \Omega^0(M)$ (a 0-form), then the differential df is a 1-form ($\in \Omega^1(M)$)

Next: generalize this to a map $d: \Omega^k(M) \to \Omega^{k+1}(M)$

Ex.: A 1-form in $\mathbb{R}^3$ can be written as $\omega = \omega_i dx^i$.

Is $\omega = df$ for some smooth f? (I.e., is ω exact?)

Yes if $\operatorname{curl} \omega = 0$, i.e., $\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0 \ \forall i,j$ (ω closed) (on a simply connected domain).

In form language: introduce 2-form

$$d\omega = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j$$

$$= \sum_{i<j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j - \sum_{i<j} \frac{\partial \omega_i}{\partial x^j} dx^i \wedge dx^j$$

$$= \sum_i \sum_{j<i} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i + \sum_i \sum_{j>i} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i$$

$$= \sum_i \sum_j \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i$$

$\Rightarrow$ ω closed if $d\omega = 0$ (then $d\,df = 0$).

Def.: For any $\omega = \sum_J{}' \omega_J dx^J$ (k-form on $\mathbb{R}^n$) we def. the $\boxed{\text{exterior derivative}}$

$$d\omega = d\left(\sum_J{}' \omega_J dx^J\right) := \sum_J{}' \underbrace{d\omega_J}_{\text{differential of } \omega_J : \mathbb{R}^n \to \mathbb{R}} \wedge dx^J = \sum_J{}' \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}$$

Properties:
- d is linear
- (Proofs: HW) For ω a k-form, η an ℓ-form, we have $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
- $d \circ d = 0$
- $U \subset \mathbb{R}^n$ open, $V \subset \mathbb{R}^m$ open, $F: U \to V$ smooth, then $F^*(d\omega) = d(F^*\omega)$
- For $f \in C^\infty(\mathbb{R}^n)$ we have $df = \frac{\partial f}{\partial x^i} dx^i$ = differential of f

We take these properties to generalize the def. to manifolds.

Thm.: The $\boxed{\text{exterior differentiation}}$ $d: \Omega^k(M) \to \Omega^{k+1}(M)$ is, for any k, uniquely def. by the following properties:

a) d is $\mathbb{R}$-linear

b) $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

c) $d \circ d = 0$

d) For $f \in C^\infty(M) = \Omega^0(M)$, df is the differential (def. by $df(X) = Xf$)

In any smooth chart, the formula above (for $\mathbb{R}^n$) holds.

Proof: See Lee's book.

(Idea: Existence: Def. by coordinate formula in each chart, then show independence of choice of chart. Uniqueness: Use properties.

We still have the important property: For $F: M \to N$ smooth, $\omega \in \Omega^k(N)$:

$$\boxed{F^*(d\omega) = d(F^*\omega)}$$