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Recall: Exterior derivative d of a k-form ω in coordinates:

For $\omega = {\sum_J}' \omega_J dx^J$ we have $d\omega = {\sum_J}' d\omega_J \wedge dx^J$

Note: There are also coordinate-independent formulas.

E.g., for any 1-form ω and X, Y vector fields:

$$d\omega(X,Y) = X\Big(\underbrace{\overbrace{\omega(Y)}^{\in C^\infty(M)}}_{p \mapsto \omega(Y)|_p = \omega_p(Y(p)) \in \mathbb{R}}\Big) - Y(\omega(X)) - \omega([X,Y]) \qquad (\in C^\infty(M))$$

Check for $\omega = u\,dv$ (any other ω is a finite sum of those, locally).

$$\Rightarrow d\omega(X,Y) = d(u\,dv)(X,Y) = du \wedge dv\,(X,Y) = du(X)\,dv(Y) - dv(X)\,du(Y)$$
$$= Xu\,Yv - Xv\,Yu.$$

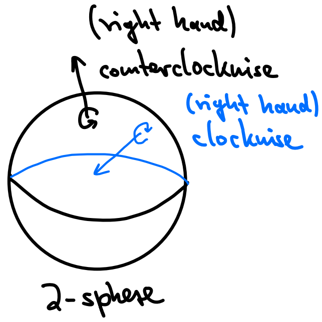
and
$$X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$
$$= X(u\,dv(Y)) - Y(u\,dv(X)) - u\,dv([X,Y])$$
$$= X(uYv) - Y(uXv) - u[X,Y]v$$
$$= Xu\,Yv + uXYv - Yu\,Xv - uYXv - uXYv + uYXv \qquad \checkmark$$

For k-forms ω: $$d\omega(X_1,\dots,X_{k+1}) = \sum_{1\le i\le k+1} (-1)^{i-1} X_i\big(\omega(X_1,\dots,\hat{X}_i,\dots,X_{k+1})\big)$$
$$+ \sum_{1\le i<j\le k+1} (-1)^{i+j}\,\omega\big([X_i,X_j], X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+1}\big),$$

where hats mean omitted arguments.

5.6 Orientation

curve: two directions



First: n-dim. vector space V

Def.: Two ordered bases $(E_1,\dots,E_n)$ and $(\hat{E}_1,\dots,\tilde{E}_n)$ of V s.t. $E_i = B_i^{\,j}\hat{E}_j$ are consistently oriented if $\det B > 0$.

An orientation $\mathcal{O}$ of V is an equivalence class $[E_1,\dots,E_n]$ of ordered bases.

V with a choice of orientation is called oriented vector space.

Note: In any vector space there are two orientations: $\underbrace{[E_1,\dots,E_n]}$ and the opposite orientation $-[E_1,\dots,E_n]$.

Any ordered basis in this orientation is "positively oriented"

Proposition: Each $0 \neq \omega \in \Lambda^n(V^*)$ (recall $\dim V = n$) determines an orientation $\mathcal{O}_\omega$ by setting

$$\mathcal{O}_\omega = [E_1,\dots,E_n] \text{ for } \omega(E_1,\dots,E_n) > 0.$$

Proof: Let $(E_1,\dots,E_n)$ and $(\tilde{E}_1,\dots,\tilde{E}_n)$ be ordered bases, B the matrix of the basis change, i.e., $\hat{E}_j = B_j^i E_i$. Then

$$\omega(\tilde{E}_1,\dots,\tilde{E}_n) = \omega(BE_1,\dots,BE_n) = (\det B)\,\omega(E_1,\dots,E_n), \text{ i.e.,}$$

$(E_1,\dots,E_n)$ and $(\hat{E}_1,\dots,\hat{E}_n)$ are consistently oriented $\iff \det B > 0$

$\iff \omega(E_1,\dots,E_n)$ and $\omega(\tilde{E}_1,\dots,\hat{E}_n)$ have the same sign.

Now: Orientation on manifolds M

Two approaches: A) via tangent spaces

B) via charts

A) A pointwise orientation is def. by choosing orientation of each T_pM.

<u>Def.</u>: • $\boxed{\text{local frame}}$: (continuous) vector fields $E_1, \dots, E_n$ on $U \subset M$ s.t. $\underbrace{(E_1|_p, \dots, E_n|_p)}_{\text{basis of } T_pM}$

• $\boxed{\text{global frame}}$: local frame on $U = M$

An $\boxed{\text{orientation on } M}$ is a continuous pointwise orientation, i.e., $\forall p \in M \; \exists$ oriented local frame (E_i) with $p \in U$ (= domain of (E_i))

Note that there are $\boxed{\text{orientable}}$ and $\boxed{\text{nonorientable}}$ manifolds.

↳ e.g., sphere ↳ e.g., Möbius strip

<u>Proposition:</u> Let M be a smooth n-manifold. Then any nonvanishing n-form ω on M determines a unique orientation on M (for which ω is positively oriented at each point).
Conversely: If M has an orientation, then there is a nonvanishing n-form on M that is positively oriented at each point.

<u>Proof:</u> (E_i) local frame on connected set U, (ε^i) dual co-frame

$\Rightarrow \omega = f\, \varepsilon^1 \wedge \dots \wedge \varepsilon^n$, with f nonvanishing (since ω nonvanishing).

$\Rightarrow \omega(E_1, \dots, E_n) = f \neq 0$ on U. Since U connected, this is always either pos. or neg.

Converse: clear (or with partition of unity on fibres).

B) Orientation in terms of charts:

- A chart is positively oriented if the coordinate frame is.
- A smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ is consistently oriented if Jacobian of $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive determinant everywhere (on $\varphi_\alpha(U_\alpha \cap U_\beta)$) and $\forall \alpha, \beta$.

<u>Proposition:</u> consistently oriented smooth atlas $\Longleftrightarrow$ orientation on M