Topology and Manifolds, Constructor University Bremen, Spring 2023
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5.7 Integration on Manifolds

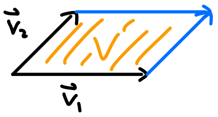
Want: coordinate independent integral on M

Idea, e.g., for $\overline{B} \subset \mathbb{R}^n$ the closed unit ball: Just integrate $f: \overline{B} \to \mathbb{R}$, e.g., $f(x) \equiv 1$ (constant fct.): $\int_{\overline{B}} f \, dV = \mathrm{Vol}(\overline{B})$, but this is clearly <u>not</u> invariant under coordinate transformations.

Better: Covector fields take at each $p \in M$ tangent vectors, linearly, i.e., the longer the tangent vector the longer the result.

Furthermore: "Signed volumes" are given, e.g. by the determinant in $\mathbb{R}^n$.

E.g.



Volume $V = \det(\vec{v}_1, \vec{v}_2)$, which is an alternating tensor

Properties we want:
- scaling one vector by λ scales the volume by λ
- $\mathrm{vol}(\vec{v}_1 + \vec{\tilde{v}}_1, \vec{v}_2, \dots) = \mathrm{vol}(\vec{v}_1, \vec{v}_2, \dots) + \mathrm{vol}(\vec{\tilde{v}}_1, \vec{v}_2, \dots)$

} multilinear

- $\mathrm{vol}(\vec{v}_1, \vec{v}_1, \dots) = 0$

$\Longrightarrow$ want alternating covariant k-tensors

Let us start 1-forms ω on $[a,b] \subset \mathbb{R}$, i.e., $\omega_t = f(t) dt$.

Then we def.
$$\boxed{\int_{[a,b]} \omega := \underbrace{\int_a^b f(t) dt}_{\text{usual Riemann (or Lebesgue) integral}}}$$

More generally, let us consider a domain of integration $D \subset \mathbb{R}^n$ (i.e., D bounded, ∂D measure 0)

Let $\omega \in \Omega^n(\bar{D})$ (n-form), i.e., $\omega = f\, dx^1 \wedge \dots \wedge dx^n$

↳ smooth (or cont. is enough)

Then we def. $$\boxed{\int_D \omega = \int_D f\, dx^1 \wedge \dots \wedge dx^n := \underbrace{\int_D f\, dx^1 \dots dx^n}_{\text{Riemann int.}} = \int_D f\, dV}$$

Important now: Invariance under coordinate transformations, or, more generally, pullbacks.

M, N oriented smooth manifolds

Def.: A (local) diffeomorphism $G: M \to N$ is called $\boxed{\text{orientation preserving}}$ if for each $p \in M$, we have that dF_p takes positively oriented bases of T_pM to positively oriented bases of $T_{F(p)}N$.

Proposition: Let $D, E \subset \mathbb{R}^n$ be open domains of integration, $G: \bar{D} \to \bar{E}$ smooth (i.e., G can be continued to a smooth map $G: \underset{\bar{D}}{\underset{\cup}{U}} \to \underset{\bar{E}}{\underset{\cup}{V}}$, with U, V open) and orientation-preserving diffeomorphism from $D \to E$, ω an n-form on $\bar{E}$. Then $$\boxed{\int_D G^*\omega = \int_E \omega.}$$

Note: $\int_D G^*\omega = -\int_E \omega$ if G orientation-reversing.

Proof: $(y^1, \dots, y^n)$ coordinates on E, $(x^1, \dots, x^n)$ coordinates on D, $\omega = f\, dy^1 \wedge \dots \wedge dy^n$

$$\Rightarrow \int_E \omega := \int_E f\, dV \underset{\substack{\uparrow \\ \text{change of variables} \\ \text{for Riemann int.}}}{=} \int_D (f \circ G)\, |\det \underbrace{DG}_{\text{Jacobian}}|\, dV \underset{\substack{\uparrow \\ G\text{ orientation} \\ \text{preserving}}}{=} \int_D (f \circ G)(\det DG)\, dV$$

$$=: \int_D (f \circ G)(\det DG)\, dx^1 \wedge \dots \wedge dx^n \underset{\substack{\nearrow \\ \text{pullback} \\ \text{formula}}}{=} \int_D G^*\omega \qquad \square$$

Next: Let M be an oriented smooth n-manifold.

First, suppose n-form ω has compact support contained in one smooth chart (U,φ) (positively oriented).

Def.: The integral of ω over M is
$$\boxed{\int_M \omega := \int_{\varphi(U)} \underbrace{(\varphi^{-1})^*\omega}_{n\text{-form on } \varphi(U)\subset\mathbb{R}^n}}$$

Proposition: $\int_M \omega$ does not depend on choice of smooth chart.

Proof: Take smooth charts (U,φ), $(\hat{U},\hat\varphi)$, s.t. $\operatorname{supp}\omega \subset U\cap\hat{U}$

$\Rightarrow$ $\hat\varphi\circ\varphi^{-1}$ orientation-preserving diffeomorphism (if both charts are pos./neg. oriented)

$$\Rightarrow \int_{\tilde\varphi(\tilde U)} (\tilde\varphi^{-1})^*\omega = \int_{\hat\varphi(\hat U\cap U)} (\tilde\varphi^{-1})^*\omega \underset{\substack{\uparrow \\ \text{previous proposition}}}{=} \int_{\varphi(\hat U\cap U)} (\hat\varphi\circ\varphi^{-1})^*(\hat\varphi^{-1})^*\omega = \int_{\varphi(\hat U\cap U)} (\varphi^{-1})^*(\tilde\varphi)^*(\hat\varphi^{-1})^*\omega$$

$$= \int_{\varphi(U)} (\varphi^{-1})^*\omega \qquad \square$$

Next
- ω compactly supported n-form on M
- $\{U_i\}$ finite open cover of $\operatorname{supp}\omega$ (with positively oriented charts (U_i,φ_i))
- $\{\psi_i\}$ a smooth partition of unity (subordinate to $\{U_i\}$)

Def.:
$$\boxed{\int_M \omega := \sum_i \int_M \psi_i\omega}$$
$$\underbrace{\int_M}_{\text{well-def. on single chart } (U_i,\varphi_i)}$$

Proposition: The def. of $\int_M \omega$ is independent of the choice of open cover or partition of unity.

Proof: Let $\{\tilde{U}_j\}$ be another finite open cover of $\operatorname{supp}\omega$ with domains of positively oriented smooth charts, and let $\{\tilde{\varphi}_j\}$ be a sbordinate partition of unity.

$$\Rightarrow \sum_i \left(\int_M \varphi_i \omega \right) = \sum_i \int_M \left(\sum_j \tilde{\varphi}_j \right) \varphi_i \omega = \sum_i \sum_j \int_M \underbrace{\tilde{\varphi}_j \varphi_i \omega}_{\substack{\text{compactly supported in a single smooth chart} \\ \Rightarrow \text{independent of chart from previous proposition}}}$$

But also: $\sum_j \left(\int_M \tilde{\varphi}_j \omega \right) = \sum_{i,j} \int_M \tilde{\varphi}_j \varphi_i \omega$, so results are the same. □

Properties (HW): · linearity, orientation reversal ($\int_{-M} \omega = -\int_M \omega$), positivity (ω pos. oriented $\Rightarrow \int_M \omega > 0$)

· diffeomorphism-invariance : $F: M \to N$ orientation-preserving diffeomorphism

$$\Rightarrow \int_M \omega = \int_N F^* \omega$$

Computing integrals in practice: Divide M into suitable $D_1, \dots, D_n$.

Example: $M = \mathbb{R}^2 \setminus \{0\}$, $\omega = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$, curve $\gamma : [0, 2\pi] \to M$, $\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$$\Rightarrow \gamma^* \omega = \frac{\cos t \, (\overbrace{d \sin t}^{= \frac{\partial \sin t}{\partial t} dt}) - \sin t \, (\overbrace{d \cos t}^{= \frac{\partial \cos t}{\partial t} dt})}{(\cos t)^2 + (\sin t)^2} = (\cos t)^2 dt + (\sin t)^2 dt = dt$$

$$\Rightarrow \int_\gamma \omega = \int_{[0,2\pi]} \gamma^* \omega := \int_{[0,2\pi]} dt = 2\pi .$$