5.9 Stokes Theorem

Thm.: Let M be an oriented smooth n-manifold with boundary and ω a compactly supported smooth $(n-1)$-form on M. Then

$$\boxed{\int_M d\omega = \int_{\partial M} \omega \qquad \text{(Stokes Theorem)}}$$

Remarks:
- $d\omega$ = exterior derivative of ω = n-form
- ∂M has orientation induced by M (Stokes orientation)
- ω on right-hand side means $i_{\partial M}^{*}\omega$ ($i_{\partial M}: \partial M \to M$ inclusion)

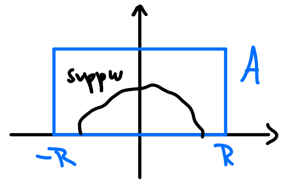
Corollary: M a compact oriented smooth manifold
- without boundary, then $\int_M d\omega = 0$ (η exact means $\eta = d\omega \Rightarrow \int_M$ exact form $= 0$)
- with boundary and ω closed (i.e., $d\omega = 0$), then $\int_{\partial M} \omega = 0$

Ex.: D a compact regular domain in $\mathbb{R}^2$, P and Q smooth fct.s. Choose $\omega = P\,dx + Q\,dy$.

Then $\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_{\partial D} (P\,dx + Q\,dy)$.

Proof of Stokes:

First case: $M = \mathbb{H}^n \Rightarrow \operatorname{supp}\omega \subset A = [-R,R]\times\dots\times[-R,R]\times[0,R]$



for some large enough R

General (n-1)-form: $\omega = \sum_i \omega_i \, dx^1\wedge\dots\wedge\widehat{dx^i}\wedge\dots\wedge dx^n$

↳ hat means omitted

$$\Rightarrow d\omega = \sum_{i=1}^n \underbrace{d\omega_i}_{\sum_{j=1}^n \frac{\partial \omega_i}{\partial x^j} dx^j} \wedge dx^1\wedge\dots\wedge\widehat{dx^i}\wedge\dots\wedge dx^n = \sum_{i=1}^n (-1)^{i-1}\frac{\partial \omega_i}{\partial x^i}\, dx^1\wedge\dots\wedge dx^n$$

$$\Rightarrow \int_{\mathbb{H}^n} d\omega = \sum_{i=1}^n (-1)^{i-1}\int_0^R\int_{-R}^R\dots\int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x)\, dx^1\dots dx^n$$

$$= \sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\dots\underbrace{\int_{-R}^R \frac{\partial \omega^i}{\partial x^i}(x)\, dx^i}_{=\ \omega^i(x)\big|_{x^i=-R}^{x^i=R} = 0 \quad (\operatorname{supp}\omega\subset A)}\, dx^1\dots\widehat{dx^i}\dots dx^n$$

$$+ (-1)^{n-1}\int_{-R}^R\dots\int_{-R}^R\underbrace{\int_0^R \frac{\partial \omega^n}{\partial x^n}(x)\, dx^n}_{\omega^n(x)\big|_{x^n=0}^{x^n=R} = -\omega^n(x^1,\dots,x^{n-1},0)}\, dx^1\dots dx^{n-1}$$

$$= (-1)^n\int_{-R}^R\dots\int_{-R}^R \omega^n(x^1,\dots,x^{n-1},0)\, dx^1\dots dx^{n-1}$$

and
$$\int_{\partial\mathbb{H}^n}\omega = \sum_{i=1}^n \int_{A\cap\partial\mathbb{H}^n} \omega_i(x^1,\dots,x^{n-1},0)\, dx^1\wedge\dots\wedge\widehat{dx^i}\wedge\dots\wedge dx^n$$

$$= \int_{A\cap\partial\mathbb{H}^n}\omega_n(x^1,\dots,x^{n-1},0)\, dx^1\wedge\dots\wedge dx^{n-1}$$

since $(x^1,\dots,x^{n-1})$ positively oriented for $\partial\mathbb{H}^n$ with even n (neg. for odd n), equality follows.

Next case: $M = \mathbb{R}^n \Rightarrow$ choose cube $A = [-R,R]^n$

By analogous computation as above, both sides are zero: $\omega^n(x)\big|_{x^n=-R}^{x^n=R} = 0$ and $\partial M = \emptyset$.

Next: ω supported in single chart (U,φ) (pos. oriented boundary chart):

$$\int_M d\omega = \int_{\mathbb{H}^n} (\varphi^{-1})^* d\omega = \int_{\mathbb{H}^n} d\left((\varphi^{-1})^*\omega\right) = \int_{\partial\mathbb{H}^n} (\varphi^{-1})^*\omega \quad \text{by computation above.}$$

Next: general case:

$$\int_{\partial M} \omega = \sum_i \int_{\partial M} \psi_i \omega = \sum_i \int_M d(\psi_i\omega) = \sum_i \int_M (d\psi_i \wedge \omega + \psi_i \, d\omega)$$

$$= \int_M d\Big(\sum_i \psi_i\Big) \wedge \omega + \int_M \Big(\sum_i \psi_i\Big) d\omega = 0 + \int_M d\omega \qquad \square$$

Note: Stokes thm. generalizes to smooth manifolds with corners. Those are allowed to have coordinate charts that map to $\overline{\mathbb{R}}^+_n = \{(x^1,\dots,x^n) \in \mathbb{R}^n : x^1 \geq 0, \dots, x^n \geq 0\}$.

E.g., triangles, squares.

(Topologically, manifolds with corners are the same as manifolds with boundary, but with smooth structures they might differ.)

6. Geometry

6.1 Symplectic Manifolds

Motivation: Hamiltonian formulation of classical mechanics
(Also: Bogoliubov transformations!)

First: Consider a finite-dimensional vector space V.

Def.: A 2-covector ω on V is called **nondegenerate** if $\hat{\omega}: V \to V^*$, def. by $\hat{\omega}(v) = v \lrcorner \omega$, is invertible.

Note: ω nondegenerate
(HW) $\Longleftrightarrow$ For each $0 \neq v \in V$ there exists $w \in V$ s.t. $\omega(v,w) \neq 0$.
(HW) $\Longleftrightarrow$ In terms of some (hence every) basis, the matrix ω_{ij} of ω is non-singular.

Def.:
- A nondegenerate 2-covector is called **symplectic tensor**.
- V with some symplectic tensor is called **symplectic vector space**.

Ex.: Let V have dim. $2n$, denote basis by $(A_1, B_1, \ldots, A_n, B_n)$, dual basis by $(\alpha^1, \beta^1, \ldots, \alpha^n, \beta^n)$.

Def. $\omega = \sum_{i=1}^{n} \alpha^i \wedge \beta^i$.

$$\Rightarrow \omega(A_i, A_j) = 0 = \omega(B_i, B_j), \quad \omega(A_i, B_j) = \delta_{ij} = -\omega(B_j, A_i).$$

Is ω nondegenerate?

- Matrix of ω $\quad (\omega_{ij}) = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}$ is non-singular.

- Alternatively: suppose $\omega(v,w)=0\ \forall w\in V$, write $v=a^iA_i+b^iB_i$

$\Rightarrow \omega(v,A_i)=-b^i,\ \omega(v,B_i)=a^i \Rightarrow v=0 \Rightarrow \omega$ nondegenerate