Recall: • A nondegenerate 2-covector is called $\boxed{\text{symplectic tensor}}$.
• V with some symplectic tensor is called $\boxed{\text{symplectic vector space.}}$

Ex.: Let V have dim. $2n$, denote basis by $(A_1, B_1, \dots, A_n, B_n)$, dual basis by $(\alpha^1, \beta^1, \dots, \alpha^n, \beta^n)$.
Def. $\omega = \sum_{i=1}^n \alpha^i \wedge \beta^i$.

$\Rightarrow \omega(A_i, A_j) = 0 = \omega(B_i, B_j)$, $\omega(A_i, B_j) = \delta_{ij} = -\omega(B_j, A_i)$.

Note: ω is nondegenerate.

This is indeed the standard basis for symplectic vector spaces.

Def.: If (V, ω) is a symplectic vector space, then a basis as in the example is called $\boxed{\text{symplectic basis.}}$ (I.e., this is a basis where $\omega = \sum_{i=1}^n \alpha^i \wedge \beta^i$.)

With a symplectic form we can def. a notion of a symplectic complement:

Def.: Let $S \subset V$ be a linear subspace of a symplectic vector space V. Then the $\boxed{\text{symplectic complement}}$ of S is def. as $S^\perp = \{v \in V : \omega(v, w) = 0 \ \forall w \in S\}$

Just as for orthogonal complements we have:

Lemma: $\dim S + \dim S^\perp = \dim V$.

Proof: Def. $\Phi: V \to S^*$ by $\Phi(v) = (v \lrcorner \omega)|_S$, i.e., $\Phi(v)(w) = \omega(v,w)$ for $v \in V$, $w \in S$.

If Φ is surjective, then $S^\perp = \ker \Phi$ has dimension

$$\dim S^\perp = \dim \ker \Phi = \dim V - \dim \operatorname{im} \Phi = \dim V - \dim S^* = \dim V - \dim S.$$

Is Φ surjective? Choose $\varphi \in S^*$, and let $\tilde{\varphi} \in V^*$ extend φ to V^*. We know $\hat{\omega}: V \to V^*$, $v \mapsto v \lrcorner \omega$ is an isomorphism, so $\exists\, v \in V$ s.t. $\hat{\omega}(v) = \tilde{\varphi} \Rightarrow \Phi(v) = \varphi$, i.e., Φ surjective. $\square$

Ex.: If $\dim S = 1$: $\omega(\lambda v, v) = 0 \ \forall v \in S, \lambda \in \mathbb{R} \Rightarrow S \subset S^\perp$

So unlike for orthogonal complements, we do not have $S \cap S^\perp = \{0\}$ here.

<u>Def.</u>: A linear subspace S is called:
- symplectic if $S \cap S^\perp = \{0\}$,
- isotropic if $S \subset S^\perp$,
- coisotropic if $S \supset S^\perp$,
- Lagrangian if $S = S^\perp$.

For now, we use symplectic complements to show:

<u>Proposition:</u> Let ω be a symplectic tensor on a vector space V. Then V has even dimension and there exists a symplectic basis for V.

Proof: Induction on $n = \dim V$. ($n = 0$ ✓)

Let $n \geq 1$. Choose some $0 \neq A_1 \in V \Rightarrow \exists\, B_1 \in V$ s.t. $\omega(A_1, B_1) \neq 0$ (ω non-degenerate).

Let $\omega(A_1, B_1) = 1$ (by multiplying B_1 with a constant).

$\Rightarrow \{A_1, B_1\}$ must be linearly independent (since ω alternating) $\Rightarrow \dim V \geq 2$.

Now: let $S = \operatorname{span}\{A_1, B_1\} \Rightarrow \dim S^\perp = n - 2$. Note that here S is symplectic: For $v \in V$, we have

$v^\perp := v - \underbrace{\omega(A_1, v) B_1 - \omega(B_1, v) A_1}_{\in S} \in S^\perp$, since $\omega(v^\perp, A_1) = \omega(v, A_1) - \omega(A_1, v)\omega(B_1, A_1) - \omega(B_1, v)\omega(A_1, A_1) = 0 = \omega(v^\perp, B_1)$.

Thus $S^\perp$ is also symplectic.

By the induction assumption, $S^\perp$ has a symplectic basis $(A_2, B_2, \dots, A_n, B_n)$.

$\Rightarrow (A_1, B_1, A_2, B_2, \dots, A_n, B_n)$ is symplectic basis for V. $\square$

<u>Corollary:</u> Let V be a $2n$-dim. vector space and $\omega \in \Lambda^2(V^*)$. Then:

$$\omega \text{ is a symplectic tensor} \iff \omega^n := \underbrace{\omega \wedge \dots \wedge \omega}_{n \text{ times}} \neq 0$$

Proof: "$\Rightarrow$" (A_i, B_i) symplectic basis, $\omega = \sum_i \alpha^i \wedge \beta^i$.

$$\Rightarrow \omega^n = \sum_I \alpha^{i_1} \wedge \beta^{i_1} \wedge \dots \wedge \alpha^{i_n} \wedge \beta^{i_n} = n! \left(\alpha^1 \wedge \beta^1 \wedge \dots \wedge \alpha^n \wedge \beta^n\right) \neq 0.$$

"$\Leftarrow$" If ω is degenerate, $\exists\ 0 \neq v \in V$ s.t. $v \lrcorner \omega = 0$.

Then $v \lrcorner \omega^n = v \lrcorner (\omega \wedge \omega^{n-1}) = (v \lrcorner \omega) \wedge \omega^{n-1} + \underbrace{(-1)\ \omega \wedge (v \lrcorner \omega^{n-1})}_{= (v \lrcorner \omega^{n-1}) \wedge \omega}$

(Recall: $i_v(\omega \wedge \eta) = (i_v \omega) \wedge \eta + (-1)^k \omega \wedge (i_v \eta)$)
k = degree of ω

$$\Rightarrow v \lrcorner \omega^n = n (v \lrcorner \omega) \wedge \omega^{n-1} = 0$$

Now we could extend v to a basis $(E_1, E_2, \dots, E_{2n})$ with $E_1 = v$, then $\omega^n(E_1, \dots, E_{2n}) = 0$, so $\omega^n = 0$, which contradicts our assumption. $\square$

<u>Def.:</u> Let (V, ω) and (W, σ) be symplectic vector spaces, then a linear map $f: V \to W$ is called **symplectic** if $f^* \sigma = \omega$, i.e., $f^* \sigma(u,v) := \sigma(fu, fv) = \omega(u,v)\ \forall u, v \in V$.

Ex.: Let $W = V$, $\sigma = \omega$, A = matrix of linear map $f: V \to V$, J = matrix of ω.

$$\Rightarrow \omega(fu, fv) = \omega(A^i_j u^j E_i, A^i_j v^j E_i) = \sum_{k,l} J_{kl}\, \varepsilon^k \wedge \varepsilon^l (A^i_j u^j E_i, A^{i_2}_{j_2} v^{j_2} E_{i_2})$$

$$= J_{i i_2} A^i_j u^j A^{i_2}_{j_2} v^{j_2} = \langle u, A^T J A v \rangle$$

If f symplectic $\overset{\downarrow}{=} \omega(u,v) = \omega(u^{i}E_{i}, v^{i_2}E_{i_2}) = J_{i i_2} u^{i} v^{i_2} = \langle u, Jv\rangle$

$$\Longrightarrow \boxed{f \text{ symplectic} \iff A^T J A = J}$$

$$\iff J^{-1}A^T J = A^{-1}$$

Note: f symplectic $\Rightarrow \det A^{-1} = \frac{1}{\det A} = \det(J^{-1}A^T J) = \det A^T = \det A$

$$\Longrightarrow |\det A| = 1$$

In fact $\det A = 1$: $\omega^n = n!\,(\alpha^1\wedge\beta^1\wedge\ldots\wedge\alpha^n\wedge\beta^n) \neq 0.$

Then $f^*\omega = \omega$ implies $f^*(\alpha^1\wedge\beta^1\wedge\ldots\wedge\alpha^n\wedge\beta^n) = (\det f)(\alpha^1\wedge\beta^1\wedge\ldots\wedge\alpha^n\wedge\beta^n)$

$$= (\alpha^1\wedge\beta^1\wedge\ldots\wedge\alpha^n\wedge\beta^n)$$

Next: Manifolds

Def.: • A 2-form ω on a smooth manifold M is called nondegenerate if ω_p is a nondegenerate 2-covector for each $p\in M$.

• A $\boxed{\text{symplectic form}}$ on M is a $\overbrace{\text{closed}}^{\text{exterior derivative} = 0}$ nondegenerate 2-form.

• M with a $\underbrace{\text{symplectic form}}_{\text{or symplectic structure}}$ is called $\boxed{\text{symplectic manifold.}}$

Note: • From the discussion above we know that $\dim M$ must be even.

• Also: If ω is a symplectic form on a $2n$-dim. manifold M, then ω^n is a nonvanishing $2n$-form, so every symplectic manifold is orientable.