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6.2 Riemannian Manifolds

For Riemannian geometry, we introduce an inner product on each tangent space.

Let M be a smooth manifold with or without boundary.

Def.: · A **Riemannian metric** on M is a smooth symmetric covariant 2-tensor field g on M that is positive definite at all $p \in M$.

· A **Riemannian manifold** is a pair (M, g).

Since 2-tensor g_p is an inner product on T_pM, we sometimes denote $g_p(v,w) := \langle v, w \rangle_g$.

In local coordinates: $g = g_{ij}\, dx^i \otimes dx^j$, g_{ij} = symmetric pos. def. matrix of smooth fct.s

Recall: For two symmetric tensors α, β we def. the symm. product as $\alpha\beta := \mathrm{Sym}(\alpha \otimes \beta)$,

with $(\mathrm{Sym}\,\alpha)(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$.

We def. $\Sigma^k(V^*)$ = space of symmetric covariant k-tensors.

$$\Rightarrow \quad g = g_{ij}\, dx^i \otimes dx^j = \frac{1}{2}\Big(g_{ij}\, dx^i \otimes dx^j + \underbrace{g_{ji}}_{= g_{ij}}\, dx^j \otimes dx^i \Big) = g_{ij}\, \underbrace{dx^i dx^j}_{\text{symm. product}}$$

Ex.: Euclidean metric $\bar{g}$ on $\mathbb{R}^n$: $\bar{g} = \delta_{ij}\, dx^i dx^j = \underbrace{(dx^1)^2}_{:= dx^1 dx^1} + \dots + (dx^n)^2$

$\Rightarrow$ Here, $\bar{g}_p(v,w) = \delta_{ij} v^i w^j = \sum_{i=1}^n v^i w^i = v \cdot w$ (dot product)

In contrast to symplectic manifolds we have:

Proposition: Every smooth manifold with or without boundary has a Riemannian metric.

Proof: Idea: Use pullback of $\bar{g}$ with coordinate charts.

Choose covering of smooth coordinate charts $(U_\alpha, \varphi_\alpha)$.

$\Rightarrow g_\alpha := \varphi_\alpha^* \bar{g}$ is a Riemannian metric ($= \delta_{ij} dx^i dx^j$ in coordinates) on U_α.

Let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, def. $g = \sum_\alpha \psi_\alpha g_\alpha$ (smooth tensor field since only finitely many nonzero terms in a neighborhood around each point).

- g is symm. by def.
- Positivity?

$$g_p(v,v) = \sum_\alpha \underbrace{\psi_\alpha(p)\, g_\alpha|_p(v,v)}_{>0 \text{ for at least one } \alpha} \quad \forall\, 0 \neq v \in T_pM$$

$\square$

With a Riemannian metric we can def., for $v,w \in T_pM$:

- length/norm $|v|_g := \langle v,v\rangle_g^{1/2} = g_p(v,v)^{1/2}$
- angles θ via $\cos\theta = \dfrac{\langle v,w\rangle_g}{|v|_g |w|_g}$
- orthogonality: $\langle v,w\rangle_g = 0$

Furthermore:

Def.: A local frame $(E_1, \dots, E_n)$ on an open neighborhood $U \subset M$ is called orthonormal frame if $(E_1|_p, \dots, E_n|_p)$ are an ONB of T_pM $\forall p \in U$.

With Gram-Schmidt every local frame can be turned into an orthonormal frame. Consequently:

Proposition: Every Riemannian manifold (M,g) has a smooth orthonormal frame in a neighborhood of each $p\in M$.

Next: Pullbacks

M, N smooth manifolds, g Riemannian metric on N, $F: M\to N$ smooth. Is F^*g a Riemannian metric on M?

Recall: $(F^*g)_p(v,w) = g_{F(p)}(dF_p(v), dF_p(w)) \quad \forall v,w\in M.$

Need to keep positivity $\Rightarrow$ need injectivity of dF_p.

Proposition: F^*g a Riemannian metric on M $\iff$ F a smooth $\underbrace{\text{immersion}}_{dF \text{ has const. rank } \dim M}$.

Ex.: $F: \mathbb{R}^2\to\mathbb{R}^3$, $(u,v)\mapsto(u\cos v, u\sin v, v)$ (helicoid)

$$\Rightarrow F^*\bar{g} = \underbrace{d(u\cos v)^2}_{:= d(u\cos v)\, d(u\cos v)} + d(u\sin v)^2 + d(v)^2$$

$$= (\cos v\, du - u\sin v\, dv)^2 + (\sin v\, du + u\cos v\, dv)^2 + dv^2$$

$$= (\cos^2 v + \sin^2 v)\, du^2 + [u^2(\sin^2 v + \cos^2 v) + 1]\, dv^2 + (-u\sin v\cos v + u\sin v\cos v)\, du\, dv$$

$$= du^2 + (u^2+1)\, dv^2$$

Ex.: Change from Cartesian to polar coordinates: $\underbrace{(x,y)}_{\text{coordinates in domain}} = \underbrace{(r\cos\varphi, r\sin\varphi)}_{\text{coordinates in codomain}}$, $F = id$

$\Rightarrow \bar{g}$ in polar coordinates is: $\bar{g} = d(r\cos\varphi)^2 + d(r\sin\varphi)^2$

$$= (\cos\varphi\, dr - r\sin\varphi\, d\varphi)^2 + (\sin\varphi\, dr + r\cos\varphi\, d\varphi)^2$$

$$= dr^2 + r^2\, d\varphi^2$$

Note: If (M,g), $(\hat{M},\tilde{g})$ Riemannian manifolds, then a Riemannian isometry is a diffeomorphism $F: M \to \hat{M}$ s.t. $F^*\tilde{g} = g$.

If such an F exists, (M,g) and $(\hat{M},\hat{g})$ are called isometric.

Riemannian geometry: Properties that are invariant under local or global isometries.

Important property:

<u>Def.:</u> A Riemannian n-manifold (M,g) is called flat if it is <u>locally</u> isometric to $(\mathbb{R}^n, \bar{g})$.

Note: • Compare this with Darboux's thm. for symplectic manifolds.

• One can show that all 1-dim. Riemannian manifolds are flat.

Ex.: Surfaces of revolution.

Let C be an embedded 1-dim. submanifold of $\{(r,z) : r>0\}$,

Let S_c be the surface of revolution generated by C.

$\gamma(t) = (a(t), b(t))$ a local parametrization of C

$\Rightarrow X(t,\theta) = (a(t)\cos\theta, a(t)\sin\theta, b(t))$ a local parametrization of S_c.

$$\begin{aligned}
\Rightarrow X^*\bar{g} &= d(a(t)\cos\theta)^2 + d(a(t)\sin\theta)^2 + d(b(t))^2 \\
&= (a'(t)\cos\theta\, dt - a(t)\sin\theta\, d\theta)^2 + (a'(t)\sin\theta\, dt + a(t)\cos\theta\, d\theta)^2 + (b'(t)dt)^2 \\
&= (a'(t)^2 + b'(t)^2)\, dt^2 + a(t)^2\, d\theta^2
\end{aligned}$$

If we parametrize $\gamma(t)$ to have speed 1, i.e., $|\gamma'(t)|^2 = a'(t)^2 + b'(t)^2 = 1$, we have

$$X^*\bar{g} = dt^2 + a(t)\, d\theta^2$$

E.g.: • unit sphere (without poles): $\gamma(t) = (\sin t, \cos t)$, $0 < t < \pi$

$$\Rightarrow X^* \bar{g} = dt^2 + \sin^2 t \, d\theta^2$$

• unit cylinder: $\gamma(t) = (1, t)$, $t \in \mathbb{R}$

$$\Rightarrow X^* \bar{g} = dt^2 + d\theta^2 \quad \Rightarrow \text{cylinder is flat!}$$

For a full discussion one needs to introduce the curvature, a local invariant quantifying deviation from flatness.

Above one can show: S_c is flat $\Leftrightarrow$ C is part of a straight line.