Next: Riemannian manifolds (M,g) as metric spaces

Idea: define metric via shortest connection with a curve.

Def.: Let $\gamma:[a,b]\to M$ be a piecewise smooth curve segment. Then the $\boxed{\text{length of } \gamma}$ is

$$L_g(\gamma)=\int_a^b |\gamma'(t)|_g\,dt. \qquad \left(\text{recall: } |\gamma'(t)|_g = g_{\gamma(t)}(\gamma'(t),\gamma'(t))^{\frac{1}{2}}\right)$$

Important property: independence of parametrization:

Proposition: Let $\gamma:[a,b]\to M$ be a piecewise smooth curve segment, and $\tilde{\gamma}$ a reparametrization, i.e., $\tilde{\gamma}=\gamma\circ\varphi$ where $\varphi:[c,d]\to[a,b]$ is a diffeomorphism. Then $L_g(\gamma)=L_g(\tilde{\gamma})$.

Proof: As in Analysis II; follows from change of variables formula.

With this we can def. the distance between points on M.

Def.: For (M,g) a connected Riemannian manifold, $p,q\in M$, we def. the $\boxed{\text{(Riemannian) distance}}$ from p to q as $d_g(p,q):=\inf_\gamma L_g(\gamma)$, where the inf is taken over all piecewise smooth curve segments from p to q.

Ex.: Euclidean space $(\mathbb{R}^n,\bar{g})$: $d_{\bar{g}}(x,y)=|x-y|$

Note: d_g turns (M,g) into a metric space. In fact, we have the following nice result:

Theorem: Let (M,g) be a connected Riemannian manifold. Then (M,g) with the Riemannian distance function d_g is a metric space whose metric topology is the same as the original manifold topology.

Proof sketch: Clear: $d_g(p,q)\geq 0$, $d_g(p,p)=0$, $d_g(p,q)=d_g(q,p)$, triangle inequality

Left to show for metric space: $d_g(p,q)>0$ if $p\neq q$.

Idea: put coordinate ball of radius ε around p, consider t_0 = infimum over all t s.t. $\gamma(t)\notin\overline{V}$ (i.e., when the curve exits $\overline{V}$).

Now use the following lemma:

Let g be a Riemannian metric on an open subset $U\subset\mathbb{R}^n$. For any compact $K\subset U$ there exist $c, C>0$ s.t. $\forall x\in K$ and $\forall v\in T_x\mathbb{R}^n$ we have $c|v|_{\bar{g}}\leq |v|_g\leq C|v|_{\bar{g}}$.

This shows $L_g(\gamma)\geq c\varepsilon>0$.

Similarly one can use the lemma to show that the metric topology is the same as the manifold topology. $\square$

Note: We thus have the notions of completeness and boundedness for Riemannian manifolds.

Corollary: Every smooth manifold with or without boundary is metrizable.

Proof:
- Connected manifold without boundary: clear.
- If not connected: Let $\{M_i\}$ be connected components, $p_i\in M_i$. If $x\in M_i, y\in M_j, i\neq j$, then $d_g(x,y):=d_g(x,p_i)+\underbrace{1}_{\text{"bridge" from } p_i \text{ to } p_j}+d_g(p_j,y)$ is a distance fct.
- If the manifold M has a boundary: Consider double of M: $D(M)=M\cup_{id}M$, with $id:\partial M\to\partial M$ the identity map, i.e., $M\sqcup M$ and identify boundary points in each copy. $D(M)$ is a manifold without boundary, and a subspace of a metrizable top. space is metrizable. $\square$

Another useful feature of a Riemannian metric: Identify elements in TM and T^*M.

Def. $\hat{g}: TM \to T^*M$ by $\hat{g}(v)(w) = g_p(v,w)$ $\forall p \in M, v \in T_pM, w \in T_pM$.

In terms of vector fields: $\hat{g}(X)(Y) = g(X,Y)$ $\forall X, Y \in \mathfrak{X}(M)$

Note: $\hat{g}$ is bijective.

In smooth coordinates: $g = g_{ij}\, dx^i dx^j \Rightarrow \hat{g}(X)(Y) = g_{ij} X^i Y^j \Rightarrow \hat{g}(X) = g_{ij} X^i dx^j$

This is usually written as $\boxed{\hat{g}(X) = X_j\, dx^j =: \underbrace{X^\flat}_{} \quad , \text{with } X_j = g_{ij} X^i}$

"X flat" (musical notation), "lowering an index"

Matrix of $\hat{g}^{-1}: T^*M \to TM$ is inverse of g_{ij}, and we denote this inverse by $\overbrace{g^{ij}}^{\text{with upper indices}}$, i.e.,

$$g^{ij} g_{jk} = g_{kj} g^{ji} = \delta^i_k .$$

Thus, for $w \in \mathfrak{X}^*(M)$: $\boxed{\hat{g}^{-1}(w) = w^i \frac{\partial}{\partial x^i} =: \underbrace{w^\#}_{} \quad , \text{with } w^i = g^{ij} w_j .}$

"w sharp"

Ex.: In the example of classical mechanics of Session 26:

If particle k has mass $m_k > 0$, Newtons eq. are $M_{ij} \ddot{q}^j(t) = F_i(q(t))$.

M is a (constant-coefficient) Riemannian metric on the configuration space Q.

$\Rightarrow$ Natural isomorphism $\hat{M}: \underbrace{TQ}_{\ni (q^i, v^i)} \to \underbrace{T^*Q}_{\ni (q^i, p_i)}$ $\Rightarrow p_i(t) = M_{ij} \dot{q}^j(t)$.

$$\Rightarrow \dot{q}^i(t) = M^{ij} p_j(t) \,,\quad \dot{p}_i(t) = -\frac{\partial V}{\partial q^i}(q(t))$$

Note: With this notation we can def. the gradient as a vector field. For $f \in C^\infty(M)$,

$$\boxed{\operatorname{grad} f := (df)^\# = \hat{g}^{-1}(df)}$$

$$\Rightarrow \langle \operatorname{grad} f, X \rangle_g = \hat{g}(\operatorname{grad} f)(X) = df(X) = Xf.$$

In smooth coordinates: $\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$

A few more notes about Riemannian manifolds:

- A generalization of Riemannian manifolds are $\boxed{\text{pseudo-Riemannian manifolds}}$, i.e., smooth manifolds with a pseudo-Riemannian metric g: a smooth symmetric 2-tensor field that is nondegenerate at each point ($\hat{g}$ is an isomorphism, or, for every $v \neq 0$ $\exists w$ s.t. $g(v,w) \neq 0$).
  By basis change, the matrix of g can be made diagonal with entries ± 1, say, r times $+1$, s time -1. The pair (r,s) is called the signature of g.
  Special case: $(r,s) = (n-1, 1)$ (or $(1, n-1)$) $\Rightarrow$ $\boxed{\text{Lorentz metrics}}$ of General Relativity.

- Let (M,g) be an oriented Riemannian n-manifold with or without boundary, $n \geq 1$. Then there is a unique smooth orientation form $\omega_g \in \Omega^n(M)$, called the $\boxed{\text{Riemannian volume form}}$, that satisfies $\omega_g(E_1, \dots, E_n) = 1$ for every local oriented orthonormal frame (E_i) of M.
  In coordinates: $\boxed{\omega_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \dots \wedge dx^n}$.

With this we can define the integral of a compactly supported continuous real-valued fct. f over M as $\int_M f \omega_g =: \int_M f\, dV_g$. E.g., for M compact, its volume is $\operatorname{Vol}(M) = \int_M \omega_g$.