Constructor University
Fall 2024

September 30, 2024

Elements of Linear Algebra

Homework 3 (covering Weeks 5 and 6)

Due on October 14, 2024, before the tutorial! Please submit on moodle.

Problem 1 [**4 points**]

Using your knowledge about the rank of a matrix, prove that if $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are both bases of a vector space V, then $n = m$.

Problem 2 [**4 points**]

Consider two vectors in $\mathbb{R}^3$,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

In the standard basis

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we can write the vectors as $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$ and $\vec{y} = y_1\vec{e}_1 + y_2\vec{e}_2 + y_3\vec{e}_3$. Now compute the determinant of the matrix

$$\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

(Note that our notation is a bit symbolic here, since we have put vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ as matrix entries; but that should not bother us.) The result should be familiar to you. Where have you encountered the resulting expression before?

Problem 3 [**4 points**]

Consider the vector space $\mathcal{P}_n$ of polynomials of degree at most n and consider the linear map $D : \mathcal{P}_n \to \mathcal{P}_n$ that maps each polynomial to its derivative. Compute the matrix of D in the basis $\{1, x, x^2, \ldots, x^n\}$ and find its determinant.

Problem 4 [**4 points**]

Let A be an invertible $n \times n$ matrix. If we know that both A and A^{-1} only have integer entries, what are possible values of $\det(A)$? *Hint: Consider how* $\det(A)$ *can be written as a function of entries of* A.

Problem 5 [4 points]

Let A be an $n \times n$ matrix with entries $a_{i,j}$ and consider the matrix B with entries $b_{i,j} = \frac{ia_{i,j}}{j}$. What is $\det(B)$ in terms of $\det(A)$?