Constructor University Fall 2024 — October 14, 2024

Elements of Linear Algebra

Homework 4 (covering Weeks 7 and 8)

Due on October 28, 2024, before the tutorial! Please submit on moodle.

Problem 1 [**4 points**]

Recall the following theorem from class. For an invertible $n \times n$ matrix A, an explicit formula for the inverse is

$$A^{-1} = \frac{1}{\det A} \mathrm{Adj} A.$$

Prove this theorem. (Hint: Use the Laplace expansion. If you need help, consult RHB Ch. 8.10, but still try to write down a rigorous proof on your own.)

Problem 2 [**4 points**]

Consider the linear system of equations

$$\begin{aligned} 3x_1 + 2x_2 + 5x_3 &= 8, \\ x_1 + 2x_2 + 2x_3 &= 5, \\ 2x_1 + 2x_2 + 3x_3 &= 7. \end{aligned}$$

Use Cramer's rule to determine the solution x_3. (Just use Sarrus' rule to quickly compute the necessary determinants.)

Problem 3 [**4 points**]

Recall the definition of the classical adjoint $\mathrm{Adj}(A)$ of an $n \times n$ matrix A from class. We assume that A is an invertible matrix.

(a) Compute the determinant of $\mathrm{Adj}(A)$ in terms of the determinant of A.

(b) Show that the adjoint of the adjoint of A is guaranteed to equal A if $n = 2$, but not necessarily for $n > 2$.

Problem 4 [**8 points**]

In class, we discussed several properties of eigenvalues. Let us exemplify them for the general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$.

(a) Compute all the eigenvalues of A. Depending on a, b, c, d, how many real and complex eigenvalues are there?

(b) Find conditions on a, b, c, d such that A is invertible, and give an explicit formula for the inverse.

(c) Compute the sum of the eigenvalues. Is it indeed equal to $a + d$, i.e., the sum of the diagonal entries?

(d) Compute the product of the eigenvalues. Is it indeed equal to $\det(A)$?

(e) Now, plug the matrix A (instead of the number λ) into the characteristic polynomial, and verify that the Cayley-Hamilton theorem holds, i.e., that this gives zero.