Constructor University
Fall 2024

October 28, 2024

Elements of Linear Algebra

Homework 5 (covering Weeks 9 and 10)

Due on November 11, 2024, before the tutorial! Please submit on moodle.

Problem 1 [**4 points**]

A *graph* is a mathematical structure consisting of a set $V = \{v_1, \dots, v_n\}$ of vertices and a set E of edges between these vertices. The *adjacency matrix* of a graph is a matrix A such that

$$A_{i,j} = \begin{cases} 1 & \text{there is an edge connecting } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}.$$

A *walk* of length k on a graph is a sequence of vertices $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ such that any two successive vertices have an edge connecting them. More precisely, for any j, v_{i_j} and $v_{i_{j+1}}$ are connected. The total number of walks from vertex v_i to vertex v_j is given by the (i, j) entry of A^k. Consider a graph with three vertices $V = \{v_1, v_2, v_3\}$ such that there are edges between v_1 and v_2, v_2 and v_3, and v_1 and v_3. Find the number of walks of length $k = 10$ starting at v_1 and ending at v_2.

Problem 2 [**4 points**]

In class, we briefly discussed the Jordan normal form. Just to get an idea how it can be useful, let us consider the matrix

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

with some $\lambda \in \mathbb{R}$. Find a general formula for J^k for any $k \in \mathbb{N}$. *Hint: Start by computing J^2, J^3, and then find the general pattern. It is also a good exercise to write down a nice clear proof of the formula using induction.*

Problem 3 [**4 points**]

Show that the matrix

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

is unitary, and compute all the eigenvalues and eigenvectors.

Problem 4 [4 points]

A general unitary 2×2 matrix can be written in the form

$$U = \begin{pmatrix} a & b \\ -e^{i\varphi}\overline{b} & e^{i\varphi}\overline{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$, and $\varphi \in [0, 2\pi)$. For matrices of this form, check explicitly the following properties of unitary matrices that we stated in class: that the columns are orthonormal, that the rows are orthonormal, and that $U^\dagger = U^{-1}$. Also compute the determinant of U, and verify that it has absolute value one. (The computation of eigenvalues and eigenvectors is a bit lengthy, so let us skip it here.)

Problem 5 [4 points]

Show that for anti-Hermitian matrices all eigenvalues are purely imaginary or zero, i.e., that they can be written as $\lambda = i\alpha$ with $\alpha \in \mathbb{R}$.

Bonus Problem [8 points]

Let us consider the Hermitian matrix

$$H = \frac{\pi}{4} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Compute e^{iH} in two different ways: first, by diagonalizing H, and second, by directly computing the power series (think about what H^2 is, and deduce from that what H^k is). Verify that the resulting matrix is unitary.