Week 3: Vector Spaces, Linear Maps, Matrices

1. MULTI Single

   Does the set of all positive reals together with the following addition and multiplication by scalar $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ form a vector space over $\mathbb{R}$ (with the scalars $c \in \mathbb{R}$)?

   $$v_1 \tilde{+} v_2 \overset{def}{=} v_1 \cdot v_2; \quad c \tilde{\cdot} v_2 \overset{def}{=} c \cdot v_2$$

   (a) $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ is not a vector space over $\mathbb{R}$
   (b) $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ is a vector space over $\mathbb{R}$

2. MULTI Single

   Is $\mathbb{Z}$, the set of all integers, a field?

   (a) Yes.
   (b) No.

3. MULTI Single

   Find a basis for $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| 7x + 2y - 5z = 0 \right\} \subset \mathbb{R}^3$.

   (a) $\begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$
   (b) $\begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$
   (c) $\begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 10 \\ 5 \\ 14 \end{bmatrix}$
   (d) $\begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$

4. MULTI Single

   Find a basis for $\left\{ \begin{bmatrix} 3a \\ -7a \\ 11a \end{bmatrix} \in \mathbb{R}^3 \middle| a \in \mathbb{R} \right\} \subset \mathbb{R}^3$.

   (a) $\begin{bmatrix} -3 \\ -7 \\ 11 \end{bmatrix}$
   (b) $\begin{bmatrix} 15 \\ -35 \\ 55 \end{bmatrix}$
   (c) $\begin{bmatrix} 51 \\ -118 \\ 187 \end{bmatrix}$

(d) $\begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix}$

5. MULTI Single

Which of the following is not a basis for the space of all cubic polynomials $P_3(\mathbb{R})$?

(a) $\mathfrak{B} = \{x^3 - x^2,\ x^2 - x,\ x - 1,\ 1\}$
(b) $\mathfrak{B} = \{x^3 + x^2 + x + 1,\ (x-6)^2,\ x - 10,\ 1\}$
(c) $\mathfrak{B} = \{x^3 - x^2,\ x^3 - x,\ x^2 - x,\ x^3 - 1\}$
(d) $\mathfrak{B} = \{x^3,\ x^2,\ x,\ 1\}$

6. MULTI Single

Which of the following functions $f : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto f(x_1, x_2)$ is linear (in the sense of linear maps as defined in class)?

(a) $f(x_1, x_2) = \sin(x_1) + \sin(x_2)$
(b) $f(x_1, x_2) = 5x_1$
(c) $f(x_1, x_2) = 7x_1x_2$
(d) $f(x_1, x_2) = (x_1)^3 + 6(x_2)^4$

7. MULTI Single

Calculate the matrix product:

$$\begin{bmatrix} 1 & 2 & 9 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = ?$$

(a) $\begin{bmatrix} -6 & 9 & 8 \\ 2 & 5 & 4 \\ 5 & 9 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} -6 & 10 & 8 \\ 2 & 5 & 4 \\ 5 & 9 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} -6 & 10 & 8 \\ 2 & 6 & 4 \\ 5 & 9 & 7 \end{bmatrix}$

(d) $\begin{bmatrix} -6 & 9 & 8 \\ 2 & 6 & 4 \\ 5 & 9 & 8 \end{bmatrix}$

8. MULTI Single

Let

$$\mathcal{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Which is the inverse of $\mathcal{R}$? (The inverse of $\mathcal{R}$ is the matrix $\mathcal{R}^{-1}$ such that $\mathcal{R}^{-1}\mathcal{R} = 1$ (the identity matrix with 1's on the diagonal, and 0's everywhere else).

(a) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(b) $\begin{bmatrix} -\cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

(c) $\begin{bmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$

(d) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

9. MULTI Single

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} B = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} C = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$$

Calculate $A \cdot B \cdot C$

(a) $\begin{bmatrix} 24 & 24 \\ 2 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 12 & 18 \\ 12 & 18 \end{bmatrix}$

(d) $\begin{bmatrix} 6 & 6 \\ 12 & 18 \end{bmatrix}$

10. MULTI Single

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} B = \begin{bmatrix} 99 & 0 \\ 99 & 99 \\ 99 & 0 \\ 99 & 99 \end{bmatrix} C = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

Which of the following is a valid matrix multiplication?

(a) $C^T \cdot B^T \cdot A^T$
(b) $A^T \cdot B^T \cdot C$
(c) $B \cdot A^T \cdot C$
(d) $A \cdot B \cdot C$

Total of marks: 10