Week 5: Kernel, range, rank-nullity theorem, matrix inverse

1. MULTI Single

   Find the inverse of $A = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{bmatrix}$.

   (a) $A^{-1} = \begin{bmatrix} -2 & -3 & 2 \\ 4 & 4 & 2 \\ 2 & 4 & -2 \end{bmatrix}$

   (b) $A^{-1} = \begin{bmatrix} -2 & -3 & 2 \\ 4 & 4 & 2 \\ 2 & -4 & -2 \end{bmatrix}$

   (c) the inverse does not exist

   (d) $A^{-1} = \begin{bmatrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{bmatrix}$

2. MULTI Single

   Find the inverse of $A = \begin{bmatrix} 3.5 & -1 & 0.5 \\ 10 & -3 & 2 \\ 2.5 & -1 & 1.5 \end{bmatrix}$.

   (a) $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & -2 \\ -1 & 1 & -2 \end{bmatrix}$

   (b) $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 2 \end{bmatrix}$

   (c) the inverse does not exist

   (d) $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$

3. MULTI Single

   Find the inverse of $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

   (a) $A^{-1} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & -0.5 & 0 \end{bmatrix}$

   (b) $A^{-1} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$

   (c) $A^{-1} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$

   (d) the inverse does not exist

4. MULTI Single

Find the kernel of $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) $\ker(A) = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$
(b) $\ker(A) = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$
(c) $\ker(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \right\}$
(d) $\ker(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

5. MULTI Single

An $n \times k$ matrix A has the following kernel:

$$\ker(A) = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

What is the dimension of its image $\dim(\operatorname{im}(A))$?

(a) $\dim(\operatorname{im}(A)) = n - 3$
(b) $\dim(\operatorname{im}(A)) = k - 1$
(c) $\dim(\operatorname{im}(A)) = k - 2$
(d) $\dim(\operatorname{im}(A)) = n$

6. MULTI Single

Given the matrix $A = B \cdot C \cdot D$ find its inverse A^{-1}.

(a) $A^{-1} = C^{-1}D^{-1}B^{-1}$
(b) $A^{-1} = D^{-1}B^{-1}C^{-1}$
(c) $A^{-1} = D^{-1}C^{-1}B^{-1}$
(d) $A^{-1} = B^{-1}C^{-1}D^{-1}$

7. MULTI Single

A linear map $D : \mathbb{R}^2 \to \mathbb{R}^2$ is given by the following matrix in the standard basis: $D_{\text{st}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. How is the map represented in the following basis: $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$?

(a) $D_{\text{new}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
(b) $D_{\text{new}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(c) $D_{\text{new}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $D_{\text{new}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

8. MULTI Single

Consider the standard basis in $\mathbb{R}^3$: $\{e_x, e_y, e_z\}$.
Which of the following matrices represents a counterclockwise rotation around the z-axis?

(a) $\mathcal{R} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix}$

(b) $\mathcal{R} = \begin{bmatrix} 1 & 0 & -\sin\varphi \\ \cos\varphi & 1 & \cos\varphi \\ \sin\varphi & 0 & 1 \end{bmatrix}$

(c) $\mathcal{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\mathcal{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}$

9. MULTI Single

Consider the space of 2×2 Hermitian Matrices $H_2(\mathbb{C})$ (the space of 2×2 matrices A with complex entries such that $A^\dagger := \overline{A}^T = A$).
Which of the following is true?

(a) $H_2(\mathbb{C})$ is a vector space over the field of complex numbers, but not over the real numbers.
(b) $H_2(\mathbb{C})$ is not a vector space over the field of real numbers or complex numbers.
(c) $H_2(\mathbb{C})$ is a vector space over the field of real numbers, but not over the complex numbers.
(d) $H_2(\mathbb{C})$ is a vector space over the field of real numbers and over the complex numbers.

10. MULTI Single

Consider the vector space $P_2(\mathbb{R}) = \{p(x) \mid p(x) \text{ is a quadratic polynomial}\}$.
Is the derivative operator $\mathcal{D} : p(x) \mapsto p'(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x} p(x)$ a linear operator? If it is, how is it represented in the standard basis $\mathfrak{B} = \{1, x, x^2\}$

Hint: You can express a polynomial $ax^2 + bx + c$ *as* $\begin{bmatrix} c \\ b \\ a \end{bmatrix}$

(a) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\mathcal{D}$ is not a linear operator

(c) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(d) $\mathcal{D}$ is a linear operator with $[\mathcal{D}]_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Total of marks: 10