Week 6: The determinant

1. MULTI Single

   A $4{\times}4$ invertible matrix A has determinant $\det(A) = \frac{1}{2}$. Find $\det(2A), \det(-A), \det(A^2)$, and $\det(A^{-1})$.

   (a) $\det(2A) = 2, \det(-A) = \frac{1}{2}, \det(A^2) = \frac{1}{2}, \det(A^{-1}) = \frac{1}{2}$
   (b) $\det(2A) = 1, \det(-A) = -\frac{1}{2}, \det(A^2) = \frac{1}{4}, \det(A^{-1}) = 2$
   (c) $\det(2A) = 1, \det(-A) = \frac{1}{2}, \det(A^2) = \frac{1}{2}, \det(A^{-1}) = \frac{1}{16}$
   (d) $\det(2A) = 8, \det(-A) = \frac{1}{2}, \det(A^2) = \frac{1}{4}, \det(A^{-1}) = 2$

2. MULTI Single

   A rotation about the y-axis by an angle θ in $\mathbb{R}^3$ is described by the matrix

   $$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}.$$

   What is $\det(R_y(\theta))$?

   (a) -1
   (b) 1
   (c) $\cos^2(\theta) - \sin^2(\theta)$
   (d) 0

3. MULTI Single

   What is the volume of the parallelopiped spanned by the vectors $v_1 = (3, 2, 1)$, $v_2 = (0, 3, 2)$, $v_3 = (0, 0, 3)$?

   (a) 12
   (b) 27
   (c) 0
   (d) 9

4. MULTI Single

   What is the determinant of the $n \times n$ matrix

   $$U = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & \cdots & 2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (n-1) & (n-1) \\ 0 & \cdots & 0 & 0 & n \end{bmatrix}.$$

   (a) n
   (b) $\dfrac{n(n+1)(2n+1)}{6}$
   (c) $n!$

(d) 0

5. MULTI Single

Consider the $n \times n$ matrix C_n with entries which simply count from 1 to n^2. For example $C_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. What is the determinant of C_n?

(a) $\det(C_1) = 1, \det(C_2) = 2, \det(C_n) = n$ for $n > 2$.
(b) $\det(C_1) = 1, \det(C_2) = -2, \det(C_n) = 0$ for $n > 2$.
(c) $\det(C_n) = (-1)^{n+1}n$
(d) $\det(C_1) = 1, \det(C_2) = -2, \det(C_3) = 0, \det(C_n) = -n^2$ for $n > 3$.

6. MULTI Single

Let $u = (2, 3, 5), v = (-1, 4, -10), w = (1, -2, -8)$ be vectors in $\mathbb{R}^3$. Use facts about the determinant to check whether u, v, w are linearly independent.

(a) The vectors are linearly independent.
(b) The vectors are not linearly independent.
(c) Cannot be determined from the information given.

7. MULTI Single

First recall that in general $\det(A+B) \neq \det(A)+\det(B)$. Now let $p, q, r, s \in \mathbb{R}$ and consider the matrices

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -r & -s \\ p & q \end{bmatrix}.$$

Compute $\det(A) + \det(B)$ and $\det(A + B)$. If these values are equal what is the common value? If not, what is the difference $\det(A + B) - (\det(A) + \det(B))$?

(a) $\det(A + B) - \det(A) - \det(B) = -rq$
(b) $\det(A + B) - \det(A) - \det(B) = rq$
(c) $\det(A) + \det(B) = \det(A + B) = 2(ps - qr)$.
(d) $\det(A + B) - \det(A) - \det(B) = -2ps$

8. MULTI Single

Consider the matrix

$$H = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

What are the cofactors $C_{1,1}$ and $C_{1,2}$? What is $\det(H)$?

(a) $C_{1,1} = 5, C_{1,2} = -2, \det(H) = 0$
(b) $C_{1,1} = 5, C_{1,2} = 2, \det(H) = 8$
(c) $C_{1,1} = 2, C_{1,2} = 2, \det(H) = 8$
(d) $C_{1,1} = 5, C_{1,2} = -2, \det(H) = 8$

9. MULTI Single

Consider the matrix

$$H = \begin{bmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{bmatrix}$$

What are the cofactors $C_{3,4}$ and $C_{4,4}$?

(a) $C_{3,4} = -27, C_{4,4} = -1$
(b) $C_{3,4} = 27, C_{4,4} = 1$
(c) $C_{3,4} = -27, C_{4,4} = 1$
(d) $C_{3,4} = 27, C_{4,4} = -1$

10. MULTI Single

Consider the matrix

$$H = \begin{bmatrix} -\lambda & 2 & 7 & 12 \\ 3 & 1-\lambda & 2 & -4 \\ 0 & 1 & -\lambda & 7 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}$$

where λ is an unknown. Find the $C_{4,4}$ cofactor and compute the determinant of the matrix.

(a) $\det(H) = \lambda^4 + 8\lambda^3 + 3\lambda + 5$
(b) $\det(H) = \lambda^4 + \lambda^3 + 6\lambda^2 + 4$
(c) $\det(H) = \lambda^4 + \lambda^3 + \lambda^2 - 5\lambda + 42$
(d) $\det(H) = (2-\lambda)C_{4,4} = \lambda^4 - 3\lambda^2 - 6\lambda^2 - 5\lambda + 42$

Total of marks: 10