Week 9: Diagonalization and Normal Matrices

1. MULTI Single

   Is the matrix
   $$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$$
   diagonalizable?

   (a) A is diagonalizable
   (b) Information given is insufficient
   (c) A is not diagonalizable

2. MULTI Single

   Is the matrix
   $$A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$
   diagonalizable?

   (a) Information given is insufficient
   (b) A is diagonalizable
   (c) A is not diagonalizable

3. MULTI Single

   Let $0 < \mu < 1$ and suppose an $n \times n$ matrix B has characteristic equation
   $$\lambda(\lambda - \mu)(\lambda - \mu^2) \cdots (\lambda - \mu^{n-1}).$$
   Is B diagonalizable?

   (a) Information given is insufficient
   (b) B is diagonalizable
   (c) B is not diagonalizable

4. MULTI Single

   Consider the vector space $\mathcal{P}_n$ of polynomials of degree n and consider the linear map $D : \mathcal{P}_n \to \mathcal{P}_n$ that maps each polynomial to its derivative. First compute the matrix of D in the standard basis $\{1, x, x^2, \ldots, x^n\}$ (you can refer to a previous exercise here). If $n > 0$, is D diagonalizable?

   (a) D is diagonalizable
   (b) D is not diagonalizable
   (c) Information given is insufficient

5. MULTI Single

   Is the matrix
   $$B = \begin{bmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{bmatrix}$$
   diagonalizable?

(a) Information given is insufficient
(b) B is not diagonalizable
(c) B is diagonalizable

6. MULTI Single

Find all values of k that make the matrix

$$A = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & k \\ 0 & 0 & 2 \end{bmatrix}$$

diagonalizable.

(a) A is diagonalizable only when $k = 0$.
(b) A is diagonalizable only when $|k| = 1$.
(c) A is diagonalizable only when $k \neq 0$.
(d) A is diagonalizable only when $|k| \neq 1$.

7. MULTI Single

Consider the 2×2 matrix

$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}.$$

Find a formula for A^k.

(a) $\begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix}$
(b) $\begin{bmatrix} 2 \cdot 5^k - 4^k & 5^k + 4^k \\ 2 \cdot 5^k - 4^k & -5^k + 2 \cdot 4^k \end{bmatrix}$.
(c) $\begin{bmatrix} 2 \cdot 5^k - 4^k & -5^k + 4^k \\ 2 \cdot 5^k - 2 \cdot 4^k & -5^k + 2 \cdot 4^k \end{bmatrix}$.
(d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8. MULTI Single

Suppose A is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose that for all k, $|\lambda_k| < 1$. Compute the limits $\lim_{p\to\infty} A^p$ and $\lim_{p\to\infty} \exp(A^p)$.

(a) $\lim_{p\to\infty} A^p = I$, $\lim_{p\to\infty} \exp(A^p) = 0$
(b) $\lim_{p\to\infty} A^p = I$, $\lim_{p\to\infty} \exp(A^p) = I$
(c) $\lim_{p\to\infty} A^p = 0$, $\lim_{p\to\infty} \exp(A^p) = I$
(d) $\lim_{p\to\infty} A^p = 0$, $\lim_{p\to\infty} \exp(A^p) = 0$

9. MULTI Single

Recall that when x, y are real numbers, we have the identity $\exp(x)\exp(y) = \exp(x+y)$. In this exercise we will investigate whether this holds for matrices. Consider the matrices

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and } Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Compute $\exp(X)\exp(Y)$ and $\exp(X+Y)$.

(a) The matrix is exponential is undefined for $\exp(Y)$.

(b) $\exp(X)\exp(Y) = \exp(X+Y) = \begin{bmatrix} e & e^2 - e \\ 0 & e^2 \end{bmatrix}$

(c) $\exp(X)\exp(Y)\exp(X+Y) = \begin{bmatrix} e & e \\ 0 & e^2 \end{bmatrix}$

(d) $\exp(X)\exp(Y) = \begin{bmatrix} e & e \\ 0 & e^2 \end{bmatrix}, \exp(X+Y) = \begin{bmatrix} e & e^2 - e \\ 0 & e^2 \end{bmatrix}$

10. MULTI Single

Consider the matrix

$$A = \begin{bmatrix} 1 & -2-i & 4\exp(i\frac{\pi}{4}) \\ 1+i & i & 2-7i \end{bmatrix}$$

What is $A^{\dagger}$?

(a) $\begin{bmatrix} 1 & 1-i \\ -2+i & -i \\ 4\exp(-i\frac{\pi}{4}) & 2+7i \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1+i \\ -2-i & i \\ 4\exp(i\frac{\pi}{4}) & 2-7i \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2-i & 4\exp(i\frac{\pi}{4}) \\ 1+i & i & 2-7i \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2+i & 4\exp(-i\frac{\pi}{4}) \\ 1-i & -i & 2+7i \end{bmatrix}$

Total of marks: 10