Week 12: QR decomposition

1. MULTI Single

   Consider the basis

$$\left\{ u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

   of $\mathbb{R}^3$. Find an orthonormal basis of $\mathbb{R}^3$ containing $u_1/\|u_1\|$ by applying Gram-Schmidt to the basis above.

   (a) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$

   (b) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \right\}$

   (c) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$

   (d) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$

2. MULTI Single

   This exercise assumes basic knowledge of integration. Consider the vector space of polynomials of degree at most 2 spanned by $\{1, x, x^2\}$. On this space, consider the inner product

$$\langle p, q \rangle := \int_0^1 p(x)q(x)dx$$

   and the corresponding norm (length)

$$\|p\| = \sqrt{\langle p, p \rangle} = \sqrt{\int_0^1 p(x)^2 dx}.$$

   Use the Gram-Schmidt procedure on the set of vectors $\{1, x, x^2\}$ to obtain an orthonormal basis that contains the vector 1.

   (a) $1, x - 1, x^2 + x - \frac{1}{3}$
   (b) $1, \sqrt{12}\Big(x - \frac{1}{2}\Big), \sqrt{180}\Big(x^2 - x + \frac{1}{6}\Big)$
   (c) The Gram-Schmidt procedure cannot be used
   (d) $1, x, x^2$

3. MULTI Single

Consider the decomposition

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{7}{5} & -\frac{1}{6} \\ 1 & \frac{14}{5} & \frac{1}{3} \\ 0 & 1 & -\frac{7}{6} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 1 \end{bmatrix}.$$

Is this a valid QR decomposition? If not, why not?

(a) The QR decomposition is not valid because columns of Q are not orthogonal.
(b) The QR decomposition is not valid because the columns of Q are not normalized.
(c) The QR decomposition is not valid because R is not lower-triangular.
(d) The QR decomposition is valid.

4. MULTI Single

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Use the Gram-Schmidt procedure to find an orthogonal matrix Q and upper triangular matrix R to obtain a QR-decomposition $A = QR$.

(a) $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ 0 & 0 & \frac{2\sqrt{2}}{3} \end{bmatrix}.$

(b) QR decomposition is not possible.

(c) $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \quad R = \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}.$

(d) $Q = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & \frac{2}{3} \end{bmatrix}.$

5. MULTI Single

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Use the Gram-Schmidt procedure to find an orthogonal matrix Q and upper triangular matrix R to obtain a QR-decomposition $A = QR$.

(a) $Q = \frac{1}{\sqrt{10}}\begin{bmatrix}1 & 3\\ 3 & -1\end{bmatrix}, \quad R = \frac{2}{\sqrt{10}}\begin{bmatrix}7 & 5\\ 0 & 1\end{bmatrix},$
(b) $Q = \frac{1}{\sqrt{10}}\begin{bmatrix}1 & 3\\ 3 & -1\end{bmatrix}, \quad R = \frac{2}{\sqrt{10}}\begin{bmatrix}5 & 7\\ 0 & 1\end{bmatrix},$
(c) $Q = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad R = \frac{2}{\sqrt{10}}\begin{bmatrix}5 & 7\\ 0 & 1\end{bmatrix},$
(d) QR decomposition is not possible.

6. MULTI Single

Which of the following statements is wrong?

(a) Every real $m \times n$ matrix with $m > n$ has a QR decomposition.
(b) Every invertible real square matrix has a QR decomposition.
(c) Only invertible real square matrices have QR decompositions.
(d) Every real square matrix has a QR decomposition.

7. MULTI Single

Consider the matrix

$$A = \begin{bmatrix}1 & 3 & 6\\ 1 & 2 & 2\\ 1 & 3 & 8\\ 1 & 2 & 4\end{bmatrix}.$$

Which of the following is a valid QR-decomposition (for 4×3 matrices)?

(a) None of the options are valid QR decompositions.
(b) $Q = \frac{1}{2}\begin{bmatrix}-1 & 1 & -1 & -1\\ -1 & 1 & 1 & 1\\ 1 & 1 & -1 & 1\\ 1 & -1 & 1 & -1\end{bmatrix}, \quad R = \begin{bmatrix}2 & 5 & 10\\ 0 & 1 & 4\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix}$
(c) $Q = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\\ 1 & 1\\ 1 & -1\end{bmatrix}, \quad R = \begin{bmatrix}2 & 5 & 10\\ 0 & 1 & 2\end{bmatrix}$
(d) $Q = \frac{1}{2}\begin{bmatrix}1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\end{bmatrix}, \quad R = \begin{bmatrix}2 & 5 & 10\\ 0 & 1 & 4\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix}$

8. MULTI Single

Consider the matrix

$$A = \frac{1}{2}\begin{bmatrix}2 & 5 & 7 & 0\\ 2 & 1 & -1 & 2\\ 2 & 5 & 11 & 8\\ 2 & 1 & 3 & -2\end{bmatrix}.$$

Note that A has a QR decomposition with

$$Q = \frac{1}{2}\begin{bmatrix}1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\end{bmatrix}, \qquad R = \begin{bmatrix}2 & 3 & 5 & 2\\ 0 & 2 & 4 & 2\\ 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 3\end{bmatrix},$$

with Q an orthogonal matrix with $\det(Q) = 1$. Use the QR decomposition to compute the determinant of A.

(a) $\det(A) = 12$
(b) $\det(A) = 6$
(c) $\det(A) = 24$
(d) $\det(A) = 32$

9. MULTI Single

Consider the plane with normal vector

$$n = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Find the orthogonal matrix Q that describes reflection of a vector on the plane with normal vector n. (Such reflections are used in the Householder construction.)

(a) $Q = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 3 & 1 & -4 \\ 4 & -4 & 8 \end{bmatrix}$
(b) $Q = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 8 & 2 & -4 \\ 4 & -6 & 7 \end{bmatrix}$
(c) $Q = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 8 & 1 & -4 \\ 4 & -4 & 7 \end{bmatrix}$
(d) $Q = \frac{1}{9}\begin{bmatrix} 1 & 10 & 4 \\ 8 & 10 & -3 \\ 4 & -1 & 7 \end{bmatrix}$

10. MULTI Single

Solve the least-square problem for

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}.$$

In other words: Consider the system of linear equations $Ax = b$, and compute the x with the smallest $\|Ax - b\|$.

(a) $x = \left(-\frac{4}{7}, \frac{5}{7}\right)$.
(b) $x = \left(\frac{1}{3}, \frac{2}{3}\right)$.

(c) $x = \Big(-4, 5 \Big)$.

(d) $x = \Big(4, -2, 3 \Big)$.

Total of marks: 10