Week 13: Singular Value Decomposition (SVD)

1. MULTI Single

Find the singular values of the matrix

$$M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

(a) The singular values don't exist because the matrix is not square.
(b) The singular values are 2 and 1.
(c) The singular values are 4 and 1.
(d) The singular values are $\sqrt{2}$ and 1.

2. MULTI Single

Find the singular values of the matrix

$$M = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

(a) The singular values are 25 and 9.
(b) The singular values are 5 and 3.
(c) The singular values don't exist because the matrix is not square.
(d) The singular values are $\sqrt{5}$ and $\sqrt{3}$.

3. MULTI Single

Consider the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$$

and the matrices

$$U = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Is the decomposition $A = U\Sigma V^T$ a valid singular value decomposition? If not, why not?

(a) The decomposition is not valid because U is not orthogonal.
(b) The SVD decomposition is valid.
(c) The decomposition is not valid as the singular values are incorrect.
(d) The decomposition is not valid because V is not orthogonal.

4. MULTI Single

Consider the matrix

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 17/10 & 1/10 & -17/10 & -1/10 \\ 3/5 & 9/5 & -3/5 & -9/5 \end{bmatrix}.$$

A has a singular value decomposition $A = U\Sigma V^*$ with

$$U = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad V^* = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Let

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Find all solutions x to the linear equations $Ax = b$.

(a) $x = \frac{1}{8}(1,1,1,1)^T$.
(b) There are no solutions to $Ax = b$.
(c) $x = \frac{1}{8}(1,1,1,1)^T + \lambda(1,-1,1,-1)$ for any $\lambda \in \mathbb{R}$.
(d) $x = \frac{1}{8}(1,1,1,1)^T + \lambda(1,-1,-1,-1)$ for any $\lambda \in \mathbb{R}$.

5. MULTI Single

Consider the matrix

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 17/10 & 1/10 & -17/10 & -1/10 \\ 3/5 & 9/5 & -3/5 & -9/5 \end{bmatrix}.$$

A has a singular value decomposition $A = U\Sigma V^*$ with

$$U = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad V^* = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

What are the rank and the nullity of A, i.e., what are the dimensions of the range and kernel of A?

(a) $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$.
(b) $\text{rank}(A) = 4$, $\text{nullity}(A) = 3$.
(c) $\text{rank}(A) = 3$, $\text{nullity}(A) = 4$.
(d) $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$.

6. MULTI Single

Consider the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

A has a singular value decomposition $A = U\Sigma V^*$ with

$$U = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 & -\sqrt{3} & \sqrt{2} \\ 2 & 0 & \sqrt{2} \\ -1 & \sqrt{3} & \sqrt{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let furthermore

$$b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve the least square problem for $Ax = b$ using the singular value decomposition of A, i.e., find the vector x that makes $\|Ax - b\|$ minimal.

(a) $x_1 = -1, x_2 = 1.$
(b) $x_1 = -\sqrt{3}, x_2 = 1.$
(c) $x_1 = -\frac{4}{3}, x_2 = -\frac{7}{3}.$
(d) $x_1 = -4, x_2 = -7.$

7. MULTI Single

Let A be an $m \times n$ matrix with $m \neq n$. Which of the following statements is false?

(a) The matrix A^*A is Hermitian.
(b) The matrix AA^* is positive semidefinite.
(c) The matrix AA^* is Hermitian.
(d) The matrix AA^* is an $n \times n$ matrix.

8. MULTI Single

Let A be an $m \times n$ matrix with $m > n$. Which of the following statements is false?

(a) A might have n non-zero singular values.
(b) Some of the singular values of A might be zero.
(c) A might have m non-zero singular values.
(d) None of the singular values of A are negative.

9. MULTI Single

Let A be an $m \times n$ matrix with rank r and singular value decomposition $A = U\Sigma V^*$ (with singular values in descending order). Let $u_1, \ldots, u_m$ be the columns of U, and $v_1, \ldots, v_n$ the columns of V. Which of the following statements is false?

(a) $\{v_{r+1}, \ldots, v_n\}$ is an orthonormal basis for $\mathrm{Ker}(A)$.
(b) $\mathrm{span}(v_{r+1}, \ldots, v_n) = \mathrm{Ker}(A)$.
(c) $\mathrm{span}(v_1, \ldots, v_r) = \mathrm{Ran}(A)$
(d) $\mathrm{span}(u_1, \ldots, u_r) = \mathrm{Ran}(A)$.

10. MULTI Single

An experiment has collected $N = 10$ two-dimensional data points, which were collected in the 2×10 matrix

$$A = \begin{bmatrix} 3 & -2 & -3 & -1 & -4 & 3 & 4 & -3 & 2 & 1 \\ 6 & -8 & -7 & -9 & -5 & 5 & 9 & -6 & 7 & 8. \end{bmatrix}.$$

Note that the averages in both rows are already zero. Use a suitable online calculator (e.g., Wolfram Alpha) to compute the singular value decomposition of A and identify the first principal component. Visualize the data for yourself in a two-dimensional coordinate system to make sure the result makes sense.

(a) The first principal component is approximately $(0.334, 0.943)$.
(b) The first principal component is approximately 23.916.
(c) The first principal component is approximately $(-0.323, 0.843)$.
(d) The first principal component is approximately $(-0.943, 0.334)$.

Total of marks: 10