Constructor University — January 25, 2025
Fall 2024

Elements of Linear Algebra
Final Exam (Make-up)

Instructions:

- The exam has 16 multiple choice questions (several answers can be correct!) and 3 longer questions. The total number of points is 118.
- For the multiple choice questions, it is sufficient to mark the final answer(s) only. (No solution steps necessary.) There are no negative points, but of course there are fewer points if wrong answers are selected, or if right answers are not selected.
- For the longer exercises 17, 18, and 19, you need to show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.
- You are free to refer to any results proven in class or the homework sheets unless stated otherwise (and unless the problem is to reproduce a result from class or the homework sheets).

Code of Academic Integrity

I pledge that the answers of this exam are my own work without the assistance of others or the usage of unauthorized material or information.

Sign to confirm that you adhere to the Academic Integrity Code:

Name: ____

Signature: ____________________________________

Matric./Student No.: __________________________

1. **(6 points)** Consider the function

$$f_\lambda(x) = x^2 + 2\lambda x - \lambda,$$

with parameter $\lambda \in \mathbb{R}$. Which of the following is true?

A. For $\lambda > 0$, the equation $f_\lambda(x) = 0$ has no real solution.
B. * For $\lambda < -1$, the equation $f_\lambda(x) = 0$ has two real solutions.
C. For $\lambda = 2$, the equation $f_\lambda(x) = 0$ has exactly one real solution.
D. * The range of $f_\lambda(x)$ is the interval $[-\lambda^2 - \lambda, \infty)$.
E. The range of $f_\lambda(x)$ is the interval $[0, \infty)$.
F. The domain of $f_\lambda(x)$ is the interval $[-1, \infty)$.

2. **(6 points)** Consider the vectors

$$a = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}.$$

Which of the following statements are true?

A. * The vectors a and b are NOT orthogonal.
B. * The cross product of a and b is $a \times b = \begin{pmatrix} -8 \\ -2 \\ 7 \end{pmatrix}$.
C. The length of a is $|a| = \sqrt{6}$.
D. The cross product of a and b is $a \times b = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$.
E. The vectors a and b are orthogonal.
F. * The length of a is $|a| = \sqrt{14}$.

3. **(4 points)** Consider the point $y = (6, 5, 1)$ and the line

$$x = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

Which of the following describes the plane that contains that line and the point y?

A. $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}.$

B. $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 8 \\ 3 \end{pmatrix}.$

C. $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}.$

D. $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$

E. $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$

F. * $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}.$

4. **(6 points)** Which of the following statements are true?

A. The vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ are a basis of $\mathbb{R}^3$.

B. If the two vectors a and b are linearly independent, then they are a basis of $\mathbb{R}^3$.

C. * The vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ are a basis of $\mathbb{R}^3$.

D. The vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ are linearly independent.

E. * The vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$ are linearly independent.

F. The vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ are a basis of $\mathbb{R}^3$.

5. **(4 points)** Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix}.$$

Calculate the matrix product AB.

A. $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

B. * $\begin{pmatrix} 2 & 7 \\ 8 & 19 \end{pmatrix}$.

C. $\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$.

D. $\begin{pmatrix} 8 & 7 \\ 20 & 19 \end{pmatrix}$.

E. $\begin{pmatrix} 2 & 4 \\ 5 & 17 \end{pmatrix}$.

F. $\begin{pmatrix} 2 & 8 \\ 7 & 19 \end{pmatrix}$.

6. **(4 points)** Which of the following describes the solution(s) to the system of linear equations

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 1, \\ x_2 + x_3 &= 2, \\ -x_2 + x_3 &= 2. \end{aligned}$$

A. The unique solution is $x = (10, 2, 7)$.

B. The system of equations has no solutions.

C. * The unique solution is $x = (-5, 0, 2)$.

D. The system of equations has infinitely many solutions $x = (8, 2, 1) + \lambda(2, 5, 2)$.

E. The unique solution is $x = (0, 0, 1)$.

F. The system of equations has infinitely many solutions $x = (1, 2, 0) + \lambda(2, 1, -1)$.

7. **(6 points)** A system of linear equations $Ax = b$ has been brought, through Gaussian elimination, into the reduced row-echelon form (in augmented matrix notation)

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & 5 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Which of the following statements are true?

A. The nullity of A is 3.

B. The nullity of A is 1.

C. * The general solution is $x = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 0 \\ -1 \end{pmatrix}$, for $\lambda, \mu \in \mathbb{R}$.

D. The general solution is $x = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$, for $\lambda \in \mathbb{R}$.

E. The general solution is $x = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix}$, for $\lambda, \mu \in \mathbb{R}$.

F. * The nullity of A is 2.

8. **(4 points)** Compute the determinant of the matrix

$$A = \begin{pmatrix} 4 & 1 & -2 \\ 2 & 1 & 4 \\ 2 & 2 & 1 \end{pmatrix}.$$

A. * $\det A = -26$.

B. $\det A = 34$.

C. $\det A = 13$.

D. $\det A = -6$.

E. $\det A = 1$.

F. $\det A = 0$.

9. **(4 points)** Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

A. $A^{-1} = \begin{pmatrix} 0 & -0.5 & 0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}.$

B. $A^{-1} = \begin{pmatrix} 0.5 & -0.5 & 0.5 \\ 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}.$

C. $A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$

D. $A^{-1} = \begin{pmatrix} 0.5 & 0 & 0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}.$

E. * $A^{-1} = \begin{pmatrix} 0 & -0.5 & 0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}.$

F. The matrix A is not invertible.

10. **(6 points)** Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Which of the following is true?

A. The matrix A has only the one eigenvalue 1.

B. * The vector $(2, 2)^T$ is an eigenvector of A.

C. * The eigenvalue 1 of A has geometric multiplicity 1.

D. * The matrix A has the two eigenvalues 1 and -1.

E. The eigenvalue 1 of A has geometric multiplicity 2.

F. The vector $(1, 2)^T$ is an eigenvector of A.

11. **(6 points)** Let A be an $n \times n$ matrix. Which of the following statements are true?

A. A has exactly n distinct eigenvalues.

B. * The determinant of A is given by the product of all eigenvalues, including their multiplicities.

C. The determinant of A is given by the sum of all eigenvalues, including their multiplicities.

D. * If $\lambda \neq 0$ is an eigenvalue of A, and A is invertible, then λ^{-1} is an eigenvalue of A^{-1}.

E. If $\lambda \neq 0$ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A as well.

F. All eigenspaces of A are one-dimensional

12. **(6 points)** Consider the matrix

$$A = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}.$$

Which of the following is true?

A. * A is normal.

B. A is Hermitian.

C. A is anti-Hermitian.

D. A is unitary.

E. * A is invertible.

F. * A is diagonalizable.

13. **(6 points)** Suppose U is a unitary $n \times n$ matrix. Which of the following is true?

A. The eigenvalues of U are either -1, 0, or 1.

B. U is positive definite.

C. * $|\det U| = 1$.

D. All eigenvalues of U are equal to 1.

E. * U is normal.

F. * $|Ux| = |x|$ for all vectors $x \in \mathbb{C}^n$.

14. **(4 points)** Compute the LU decomposition of the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 6 \end{pmatrix}$$

such that all diagonal entries of L are one. What are the diagonal entries of U?

A. U has diagonal entries $-1, 2$.
B. U has diagonal entries $-1, -2$.
C. U has diagonal entries $-1, 1$.
D. U has diagonal entries $1, 0$.
E. U has diagonal entries $1, 1$.
F. * U has diagonal entries $1, -2$.

15. **(4 points)** Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 6 \\ 1 & 2 & 2 \\ 1 & 3 & 8 \\ 1 & 2 & 4 \end{pmatrix}.$$

Which of the following is a valid QR-decomposition (for 4×3 matrices)?

A. * $Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & 5 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$

B. None of the options are valid QR decompositions.

C. $Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & 5 & 10 \\ 0 & 1 & 2 \end{pmatrix}.$

D. $Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 4 \\ 2 & 5 & 10 \end{pmatrix}.$

E. $Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & 5 & 10 \\ 0 & 1 & 2 \end{pmatrix}.$

F. $Q = \frac{1}{2}\begin{pmatrix} -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & 5 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$

16. **(6 points)** Consider the $m \times n$ matrix A and its singular value decomposition $A = U\Sigma V^*$. Which of the following statements are true?

A. Σ is a Hermitian $m \times n$ matrix.

B. * V is a unitary $n \times n$ matrix.

C. * U is a unitary $m \times m$ matrix.

D. The determinant of A is the sum of all singular values squared.

E. * All singular values of A are positive or zero.

F. U is a unitary $n \times n$ matrix.

17. **(12 points)**

Find the general solution to the system of linear equations

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 2, \\ 2x_1 + 6x_2 - x_4 &= 1. \end{aligned}$$

(Here, you need to write down all steps of your solution in order to receive full points.)

Solution: We use Gaussian elimination in the augmented matrix notation. We find

$$\begin{aligned} &\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 2 & 6 & 0 & -1 & 1 \end{array}\right) \\ -2\mathrm{R1} + \mathrm{R2} \to \mathrm{R2}: \quad &\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 0 & 0 & -2 & -3 & -3 \end{array}\right) \\ \frac{1}{2}\mathrm{R2} + \mathrm{R1} \to \mathrm{R1} \ \text{ and} -\frac{1}{2}\mathrm{R2} \to \mathrm{R2} \quad &\left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \end{array}\right) \end{aligned}$$

Then we can introduce two extra zero rows, which yields:

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

From this reduced row-echelon form we can immediately read off a parametrization of the general solution, namely

$$x = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \\ -1 \end{pmatrix}.$$

18. **(12 points)**

Compute all eigenvalues, singular values, and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(Here, you need to write down all steps of your solution in order to receive full points.) Is the matrix A diagonalizable? (Here, you get full point only if you justify your answer correctly.)

Solution: To find the eigenvalues we need to solve $\det(A - \lambda) = 0$. Here, we find

$$A - \lambda = \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 2 & 0 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}.$$

and a Laplace expansion yields

$$\det \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 2 & 0 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = \lambda^4.$$

Hence, there is only one eigenvalue $\lambda = 0$. The eigenvectors are solutions to

$$Ax = 0 \quad \Rightarrow \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, the solution is

$$x = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for any $\lambda \neq 0$. To find the singular values, we compute

$$A^T A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

Hence, the singular values are 3, 2, 1, and 0.

19. **(12 points)** Diagonalize the matrix

$$A = \begin{pmatrix} 11 & -4 \\ -2 & 13 \end{pmatrix}.$$

(Here, you need to write down all steps of your solution in order to receive full points.)

Solution: The characteristic equation reads

$$0 = \det(A - \lambda) = \det \begin{pmatrix} 11-\lambda & -4 \\ -2 & 13-\lambda \end{pmatrix} = (11-\lambda)(13-\lambda) - 8 = \lambda^2 - 24\lambda + 135.$$

Hence the eigenvalues are

$$\lambda_\pm = 12 \pm \sqrt{144 - 135} = 12 \pm \sqrt{9} = 12 \pm 3,$$

i.e., $\lambda_+ = 15$ and $\lambda_- = 9$.

For $\lambda_+ = 15$, any eigenvalue is solution to

$$0 = \begin{pmatrix} 11-15 & -4 \\ -2 & 13-15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Hence, we can choose for example $x_+ = (1, -1)^T$ as a normalized eigenvector.

For $\lambda_- = 9$, any eigenvalue is solution to

$$0 = \begin{pmatrix} 11-9 & -4 \\ -2 & 13-9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Hence, we can choose for example $x_+ = (2, 1)^T$ as a normalized eigenvector.

The diagonalizing matrix is thus

$$V = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix},$$

and its inverse is

$$V^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}.$$

Thus, a complete diagonalization reads

$$A = \begin{pmatrix} 11 & -4 \\ -2 & 13 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} 15 & 0 \\ 0 & 9 \end{pmatrix}}_{\Lambda} \underbrace{\frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}}_{V^{-1}}.$$