Jacobs University Bremen
Fall 2022

Calculus and Linear Algebra I
Final Exam
Time: 120 minutes

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Name: ____________________ Matriculation ID: ____________________

**INSTRUCTIONS**

- Make sure to write your name and ID on the first page and every page thereafter.
- The question booklet consists of **14 pages**. Make sure you have all of them.
- Keep quiet during the exam. For assistance, raise your hand and a proctor will come to see you.
- Answer the questions in the spaces provided after each question. If you run out of room for an answer, continue on the back of the page.
- The mark of each question is printed next to it.
- Use of mobile phones or other unauthorized electronic devices or material in the exam room is prohibited. No mathematical calculators are allowed during the exam.
- Make sure you read and sign the **Declaration Of Academic Integrity** shown below.

**Declaration of Academic Integrity**

**By signing below, I pledge that the answers of this exam are my own work without the assistance of others or the usage of unauthorized material or information.**

**Signature:** ..........................................................................

Good luck!

Dr. Stephan Juricke, Prof. Dr. Sören Petrat

| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Points | 6 | 6 | 6 | 6 | 4 | 4 | 4 | 6 | 4 | 4 | 4 | 6 | 25 | 25 | 110 |
| Score | | | | | | | | | | | | | | | |

1. **(6 points)** Let $p(x)$ be a polynomial of degree $n$ with **real** coefficients, i.e., $p(x) = \sum_{k=0}^{n} c_k x^k$, with $c_k \in \mathbb{R}$ for all $k = 0, \ldots, n$, and $c_n \neq 0$. Which of the following is true?

   A. All roots are real numbers.

   B. The roots can be real or complex numbers.

   C. If $z$ is a root, then its complex conjugate $z^*$ is also a root.

   D. If $n$ is even, then $p(x)$ must have one real root.

   E. $p(x)$ factorizes as $p(x) = a_n(x - z_1)(x - z_2)\cdots(x - z_n)$ with all $z_k$ real.

   F. $p(x)$ factorizes as $p(x) = a_n(x - z_1)(x - z_2)\cdots(x - z_n)$ and $a_n(-z_1)(-z_2)\cdots(-z_n) = c_0$.

2. **(6 points)** Consider the polynomial

$$p(x) = x^2 - 2\lambda x + 1,$$

   with some parameter $\lambda \in \mathbb{R}$. Which of the following is true?

   A. For any $-1 \leq \lambda \leq 1$ the roots are real numbers.

   B. For any $\lambda \leq -1$ and $\lambda \geq 1$ the roots are real numbers.

   C. For any $-2 \leq \lambda \leq 2$ the roots are real numbers.

   D. For $\lambda > 0$ the roots are positive, and for $\lambda < 0$ the roots are negative.

   E. For $\lambda = \frac{1}{13}$, we have $p(x) > 0$ for all $x \in \mathbb{R}$.

   F. For $\lambda = \frac{1}{13}$, we have $p(x) < 0$ for all $x \in \mathbb{R}$.

3. **(6 points)** Consider the function

$$f(x) = x \sin\left(\frac{1}{x}\right) \quad \text{for } x \neq 0,$$

   and $f(0) := 0$. Which of the following is true?

   A. $f$ is continuous at 0.

   B. $f$ is not continuous at 0.

   C. $|f(x)| \leq |x|$.

   D. $\lim_{x\to 0} f(x) = 0$.

   E. $\lim_{x\to 0} f(x) = 1$.

   F. $\lim_{x\to 0} f(x) = -1$.

4. **(6 points)** What are necessary or sufficient conditions for $f : [a, b] \to \mathbb{R}$ to have a minimum at $c \in (a, b)$?
   A. $f'(x) = 0$ for all $x \in (a, b)$.
   B. $f'(c) = 0$.
   C. $f(c) = 0$.
   D. $f''(c) > 0$.
   E. $f''(c) < 0$.
   F. $f''(c) \geq 0$.

5. **(4 points)** We want to build a fence around a rectangular field. 500 metres of fencing material are available and the field is on one side bounded by a building so that this side won't need any fencing. What is the largest area $A$ that can be fenced in?
   A. None of the given options.
   B. $A = 500m^2$
   C. $A = 50000m^2$
   D. $A = 31250m^2$
   E. $A = 12500m^2$
   F. $A = 25000m^2$

6. **(4 points)** What is the derivative of $f(x) = \sin(e^{2cx^2})$ with respect to $x$, where $c \in \mathbb{R}$ is some constant?
   A. $f'(x) = e^{2cx^2} \cos(e^{2cx^2})$
   B. $f'(x) = 2cxe^{2cx^2} \sin(e^{2cx^2})$
   C. $f'(x) = 4cxe^{2cx^2} \cos(e^{2cx^2})$
   D. $f'(x) = \cos(4cx^2 e^{2cx^2})$
   E. $f'(x) = cxe^{2cx^2} \cos(e^{2cx^2})$
   F. $f'(x) = xe^{2x^2} \cos(e^{2cx^2})$

7. **(4 points)** Compute the integral

$$\int_0^1 xe^x \, \mathrm{d}x.$$

A. $e$.
B. $2e$.
C. $e - 1$.
D. $e^x$.
E. 2.
F. 1.

8. **(6 points)** Consider the improper integral

$$\int_1^\infty \frac{1}{x^\alpha} \, \mathrm{d}x$$

for different parameters $\alpha \in \mathbb{R}$.

A. For $\alpha = 1$ the improper integral is infinite.
B. For $\alpha = 1$ the improper integral is finite and its value is 1.
C. For $\alpha = 1$ the improper integral is finite and its value is $\ln(1 + \alpha)$.
D. For $\alpha > 1$ the improper integral is infinite.
E. For $\alpha > 1$ the improper integral is finite and its value is $\frac{1}{\alpha - 1}$.
F. For $\alpha > 1$ the improper integral is finite and its value is $\ln \alpha$.

9. **(4 points)** What is the solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3yt$$

with initial condition $y(0) = 1$?

A. $y(t) = e^t$.
B. $y(t) = e^{-t}$.
C. $y(t) = e^{-\frac{3}{2}t^2}$.
D. $y(t) = e^{-t^2}$.
E. $y(t) = e^{-3t}t + 1$.
F. $y(t) = e^{-3t}t$.

10. **(4 points)** What is the scalar product between $u = (2, -3, 1)^T$ and $v = (1, 4, 5)^T$?

A. $u \cdot v = 5$

B. $u \cdot v = -3$

C. $u \cdot v = -5$

D. $u \cdot v = 2$

E. $u \cdot v = -6$

F. $u \cdot v = -10$

11. **(4 points)** Calculate $A^T \cdot B \cdot C$ with

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & -3 \\ 1 & 2 \end{bmatrix}$$

A. $\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$

B. $\begin{bmatrix} 4 & -10 \\ -4 & 10 \end{bmatrix}$

C. $\begin{bmatrix} 4 & 10 \\ 4 & 10 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 4 \\ 4 & -2 \end{bmatrix}$

E. $\begin{bmatrix} 10 & -10 \\ 4 & -4 \end{bmatrix}$

F. $\begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix}$

12. **(6 points)** Which of the following is true?

A. $\{b_1, ..., b_n\}$ with integer $n$ is a basis $\implies$ all $b_i$, $i = 1, ..., n$ are linearly independent.

B. $u \cdot v = 0 \implies u$ is parallel to v.

C. $u$ is perpendicular to $v \implies u \cdot v$ does not exist.

D. $u \times v = 0 \implies u$ and $v$ are linearly independent.

E. $u, v$ are linearly dependent $\implies au + vb = 0$ only if $a, b = 0$.

F. $u, v$ are linearly independent $\implies au + vb = 0$ only if $a, b = 0$.

13. **(25 points)**

We consider the function

$$f(x) = \frac{e^x}{x-1}.$$

2 (a) What is the domain of the function?

3 (b) What are the intercepts with the $x$-axis and with the $y$-axis?

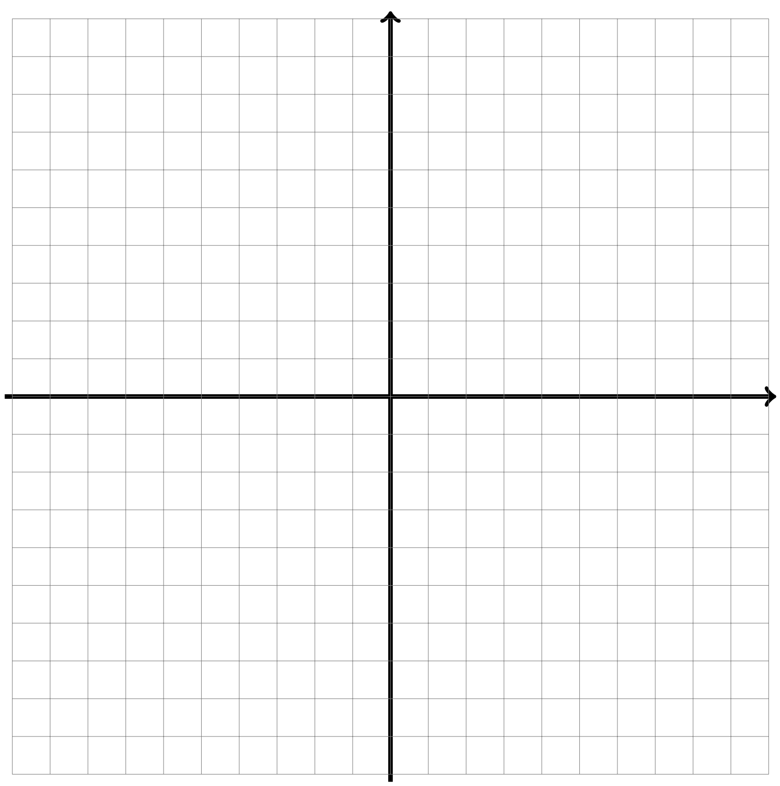
2 (c) What are the horizontal asymptotes?

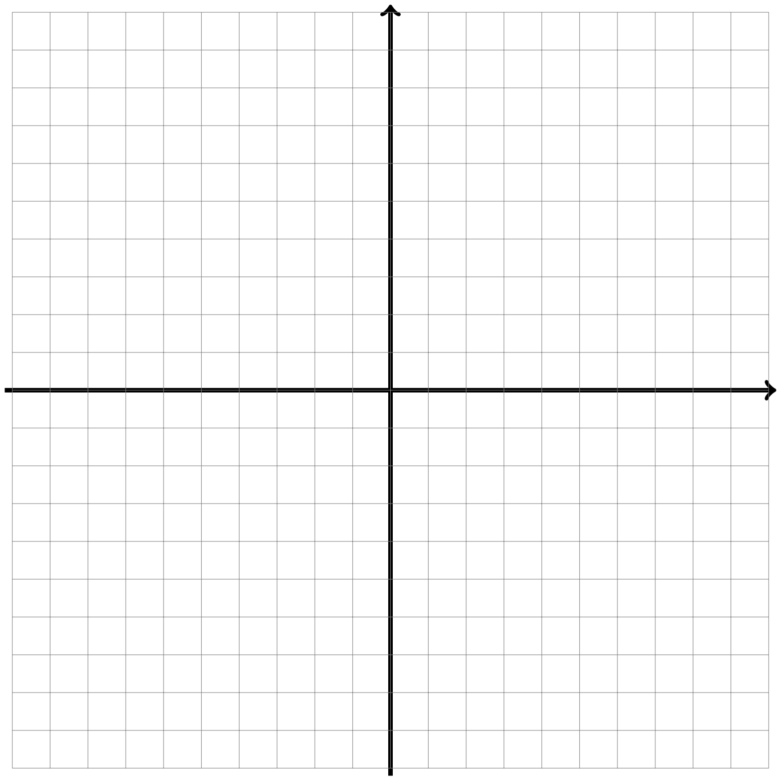
2 (d) What are the vertical asymptotes?

6 (e) Compute and analyze the first derivative. In which intervals is the function increasing or decreasing, what are the local minima or maxima?

5 (f) Compute and analyze the second derivative. In which intervals is the function concave up or concave down, what are the points of inflection?

5 (g) Sketch the function. Your drawing needs to include all the qualitative features of the graph discussed in the questions above.





14. **(25 points)**

4 (a) For a matrix $A \in M(n \times n)$, give one equivalent statement to "The inverse of $A$, i.e., a matrix $A^{-1}$ such that $A^{-1}A = \mathbb{1}$, exists".

10 (b) Find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$ using row operations as in the lecture.

4 (c) Solve the equation $Ax = b$ with $b = (1, 2, 1)^T$.

3 (d) What are the values for the Rank and Nullity of $A$? Briefly explain your answer.

4 (e) Now consider the matrix $B = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix}$, i.e., we have changed the last column in $A$.

What are the values for the Rank and Nullity of $B$? Briefly explain your answer.