Mock Exam
Calculus and Linear Algebra I
Fall 2022

Dr. Stephan Juricke, Prof. Soeren Petrat

1. **(6 points)** We consider the function

$$f(x) = e^{2x+1}.$$

Which of the following is true?

A. The domain of f is $[0, \infty)$.
B. * The domain of f is $\mathbb{R}$.
C. The domain of f is the set of all x with $x > -\frac{1}{2}$.
D. * The inverse function is $f^{-1}(y) = \frac{1}{2}\ln(y) - \frac{1}{2}$.
E. The inverse function does not exist, since f is not invertible.
F. The inverse function is $f^{-1}(y) = e^{-2y-1}$.

2. **(6 points)** In class we discussed the Extreme Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous, then f assumes its minimum and maximum. Which of the following is true?

A. * The theorem applies to $f : [0, 4] \to \mathbb{R}, f(x) = e^{\sin(x)}$.
B. The theorem applies to $f : (0, 1) \to \mathbb{R}, f(x) = \frac{1}{x}$.
C. * The theorem applies to $f : [1, 2] \to \mathbb{R}, f(x) = \frac{1}{x}$.
D. The theorem applies to $f : \mathbb{R} \to \mathbb{R}, f(x) = e^x$.
E. The theorem applies to $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{x^2+1}{x^2+2}$.
F. * The theorem applies to $f : [0, 1] \to \mathbb{R}, f(x) = \frac{x^2+1}{x^2+2}$.

3. **(6 points)** Consider the polynomial

$$p(x) = 2x^2 - 12x + 20.$$

Which of the following is true?

A. * The roots of p are $3 + i$ and $3 - i$.
B. The roots of p are $3 + i$ and $3 + i$.
C. The roots of p are $2 + i$ and $1 - 2i$.
D. The roots of p are $1 + 3i$ and $1 - 3i$.
E. $p(x)$ does not have any (possibly complex) roots.
F. * $p(x) > 0$ for all $x \in \mathbb{R}$.

4. **(6 points)** Consider the function

$$f(x) = \frac{\sin(x)}{x} + \frac{x}{x^2 - 1}.$$

Which of the following is true?

A. * The line $y = 0$ is a horizontal asymptote of f.
B. The line $y = 1$ is a horizontal asymptote of f.
C. f has a vertical asymptote at 0.
D. * f has a vertical asymptote at 1.
E. * f has a vertical asymptote at -1.
F. $f(0) = 0$.

5. **(6 points)** If the derivative f' of $f : [0,1] \to \mathbb{R}$ at x_0 exists, which of the following statements is true?
   A. * f is continuous at x_0.
   B. f'' exists at x_0.
   C. f is differentiable on $[0,1]$.
   D. * $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.
   E. f' is continuous at x_0.
   F. f is a polynomial.

6. **(4 points)** What is the derivative of $f(x) = \cos\left(\frac{\sqrt{e^x+a}}{2}\right)$ with respect to x, where $a \in \mathbb{R}$ is some constant?
   A. $f'(x) = e^x \frac{\cos(\frac{\sqrt{e^x+a}}{2})}{\sqrt{e^x+a}}$
   B. $f'(x) = -e^x \sin(\frac{\sqrt{e^x+a}}{2})$
   C. * $f'(x) = -e^x \frac{\sin(\frac{\sqrt{e^x+a}}{2})}{4\sqrt{e^x+a}}$
   D. $f'(x) = e^x \frac{\sin(\frac{\sqrt{e^x+a}}{2})}{\sqrt{e^x+a}}$
   E. $f'(x) = -2e^x \frac{\sin(\frac{\sqrt{e^x+a}}{2})}{x\sqrt{e^x+a}}$
   F. $f'(x) = -2e^x \frac{\tan(\frac{\sqrt{e^x+a}}{2})}{x\sqrt{e^x+a}}$

7. **(4 points)** We need to build a box whose base length L is 3 times its base width W. The height is given by H. Different material is used to build the top and bottom. For the top it costs 10 EUR per square metre and for the sides it costs 6 EUR per square metre. The box should have a volume of 50 cubic metres. What is the value of the width of the base W that minimize the cost to build the box?

   A. $W = (3/20)^{1/3}$
   B. $W = (20/5)^{1/3}$
   C. $W = (20/9)^{1/3}$
   D. $W = (-9/20)^{1/3}$
   E. $W = (5/3)^{1/3}$
   F. * $W = (20/3)^{1/3}$

8. **(4 points)** Compute the integral

$$\int_0^1 x^2 \sin(x^3 + 1)\,\mathrm{d}x.$$

   Hint: Substitution.

   A. $\frac{1}{3}\big(\cos(1) - 1\big)$.
   B. $\cos(2) - \cos(1)$.
   C. * $\frac{1}{3}\big(\cos(1) - \cos(2)\big)$.
   D. $\sin(2)$.
   E. $\sin(1)$.
   F. $\tan(2) - \tan(1)$.

9. **(4 points)** What is the solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y + 1$$

with initial condition $y(0) = 1$?

A. * $y(t) = 2e^t - 1$.
B. $y(t) = 2e^{t+1} - 1$.
C. $y(t) = 2e^{t^2} - 1$.
D. $y(t) = \frac{1}{t+1}$.
E. $y(t) = \frac{2}{t+1}$.
F. $y(t) = \frac{1}{(t+1)^2}$.

10. **(6 points)** Which of the following computations are possible if A is an $(n \times m)$ matrix, B an $(n \times m)$ matrix, and C an $(n \times r)$ matrix, with $n \neq m \neq r$ (positive) integers.

A. AB.
B. * AB^TC.
C. AC^T.
D. ACB.
E. CA.
F. * A^TC.

11. **(4 points)** Find the inverse of $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$.

A. $A^{-1} = \begin{bmatrix} -2 & -3 & 2 \\ -1 & -2 & -1 \\ 2 & -2 & -2 \end{bmatrix}$

B. * $A^{-1} = \begin{bmatrix} 2 & 3 & -1 \\ -1 & -2 & 1 \\ -2 & -2 & 1 \end{bmatrix}$

C. $A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ 1 & -2 & -1 \\ 2 & -2 & 2 \end{bmatrix}$

D. $A^{-1} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & -2 \\ -2 & -2 & 1 \end{bmatrix}$

E. $A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -1 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

F. $A^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

12. **(4 points)** What is the cross product between $u = (2, -3, 1)^T$ and $v = (1, 4, 5)^T$?

A. $u \times v = (19, 9, 10)$

B. $u \times v = (-9, -19, 11)$

C. * $u \times v = (-19, -9, 11)$

D. $u \times v = (11, 19, -9)$

E. $u \times v = (10, 9, -19)$

F. None of the above.

13. **(25 points)**

We consider the function

$$f(x) = \frac{3x^2 - 1}{x^2 - 1}.$$

2 (a) What is the domain of the function?

3 (b) What are the intercepts with the x-axis and with the y-axis?

2 (c) What are the horizontal asymptotes?

2 (d) What are the vertical asymptotes?

6 (e) Compute and analyze the first derivative. In which intervals is the function increasing or decreasing, what are the local minima or maxima?

5 (f) Compute and analyze the second derivative. In which intervals is the function concave up or concave down, what are the points of inflection?

5 (g) Sketch the function. Your drawing needs to include all the qualitative features of the graph discussed in the questions above.

Solution:

(a) The domain of the function is $D(f) = \{x \in \mathbb{R} : x \neq -1 \text{ and } x \neq 1\}$.

(b) The intercept with the y-axis is at $f(0) = \frac{-1}{-1} = 1$. For the intercept with the x-axis, we need to solve $f(x) = 0$. Here, $f(x) = \frac{3x^2-1}{x^2-1} = 0$ if and only if $3x^2 - 1 = 0$, i.e., the intercepts with the x-axis are at $\pm\frac{1}{\sqrt{3}}$.

(c) f has one horizontal asymptote at $y = 3$, since $\lim_{x\to\infty} f(x) = 3$ and $\lim_{x\to-\infty} f(x) = 3$ as well.

(d) f has two vertical asymptotes: at $x = -1$ and $x = 1$, since there f diverges.

(e) The first derivative is

$$f'(x) = \frac{-4x}{(x^2 - 1)^2},$$

which follows from using the quotient rule (or product rule). We find thus that $f'(x) = 0$ at $x = 0$. Furthermore:

- f increases for $x < 0$ and decreases for $x > 0$.
- At $x = 0$ there is thus a local maximum.

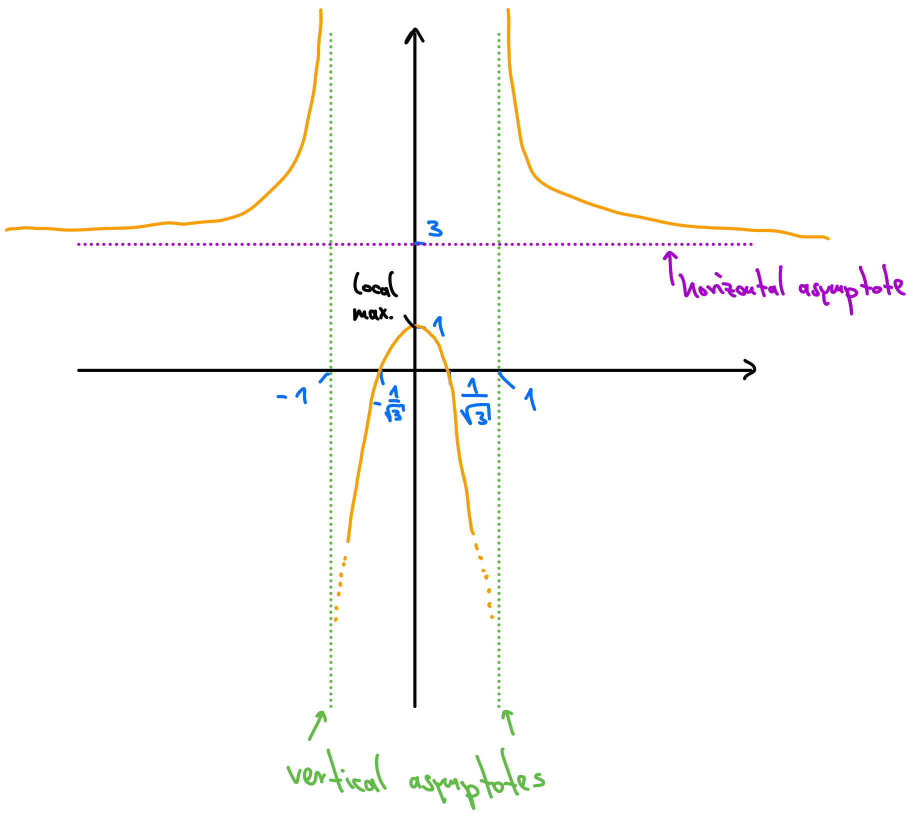
(f) The second derivative is

$$f''(x) = \frac{12x^2 + 4}{(x^2 - 1)^3},$$

which follows from using the quotient rule (or product rule). We find thus that $f''(x) = 0$ has no solution, i.e., there are no points of inflection. However, the sign of f'' changes at -1 and 1, and thus

- f is concave up in the interval $(-\infty, -1)$,
- f is concave down in the interval $(-1, 1)$,
- f is concave up in the interval $(1, \infty)$.

(g)



14. **(25 points)**

A specific plane in $\mathbb{R}^3$ is spanned by the three points $P = (2, 0, 1)$, $Q = (2, -2, 1)$ and $R = (3, 3, -1)$.

7 (a) Find two linearly independent vectors that are parallel to the plane. Briefly explain why your two vectors are linearly independent from each other.

9 (b) What method can you use to find a third vector that is orthogonal to two such linearly independent vectors? Using the vectors in (b), find a vector that is orthogonal to the plane.

9 (c) Write down the equation of the plane in normal form.

Solution:

(a) Assuming that the three points are not in a line (which we can check later), we compute the vector between two of them (using the vector representation of these points) and repeat this for a different pair of points, e.g.:

$$v_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

and

$$v_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$$

(other combinations are also possible).

We can check for their linear independence by, for example, computing the cross product between them:

$$\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If the two vectors were parallel to each other (hence, linearly dependent) the cross product would have to be the zero vector. This is not the case, so they are linearly independent. Other arguments would also hold, e.g., using the actual definition of linear independence, i.e., that one vector is not a multiple of the other. This cannot be the case, as one vector has zeros in two locations that are not zero in the other, so multiplication with a constant cannot transform from one to the other.

(b) We have to use the cross product. We already did this in (a), so we can simply use that vector:

$$\begin{pmatrix}4\\0\\2\end{pmatrix}$$

This is a vector orthogonal to the plane.

(c) For the normal form, we need one vector on the plane and a normal vector. We can, for example, use P and the vector from (b) and build the scalar product between the normal vector and a vector pointing from P to any other point (x, y, z) on the plane:

$$\left(\begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix}\right) \cdot \begin{pmatrix}4\\0\\2\end{pmatrix} = 0$$

So we get:

$4(x-2) + 0(y-0) + 2(z-1) = 0 = 4x + 2z - 10$

As a test, we can check if all three points P, Q and R are on this plane by inserting their coordinates and we'll see that it works.