Jacobs University Bremen
Fall 2022

JACOBS UNIVERSITY

Calculus and Linear Algebra I
Final Exam
Time: 120 minutes

Name: ______________________ Matriculation ID: ______________________

INSTRUCTIONS

- Make sure to write your name and ID on the first page and every page thereafter.
- The question booklet consists of **14 pages**. Make sure you have all of them.
- Keep quiet during the exam. For assistance, raise your hand and a proctor will come to see you.
- Answer the questions in the spaces provided after each question. If you run out of room for an answer, continue on the back of the page.
- The mark of each question is printed next to it.
- Use of mobile phones or other unauthorized electronic devices or material in the exam room is prohibited. No mathematical calculators are allowed during the exam.
- Make sure you read and sign the **Declaration Of Academic Integrity** shown below.

Declaration of Academic Integrity

By signing below, I pledge that the answers of this exam are my own work without the assistance of others or the usage of unauthorized material or information.

Signature: ....

Good luck!
Dr. Stephan Juricke, Prof. Dr. Sören Petrat

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
Points	6	6	6	6	4	4	4	6	4	4	4	6	25	25	110
Score															

1. **(6 points)** Let $p(x)$ be a polynomial of degree n with **real** coefficients, i.e., $p(x) = \sum_{k=0}^{n} c_k x^k$, with $c_k \in \mathbb{R}$ for all $k = 0, \ldots, n$, and $c_n \neq 0$. Which of the following is true?

   A. All roots are real numbers.
   B. The roots can be real or complex numbers.
   C. If z is a root, then its complex conjugate z^* is also a root.
   D. If n is even, then $p(x)$ must have one real root.
   E. $p(x)$ factorizes as $p(x) = a_n(x - z_1)(x - z_2)\cdots(x - z_n)$ with all z_k real.
   F. $p(x)$ factorizes as $p(x) = a_n(x - z_1)(x - z_2)\cdots(x - z_n)$ and $a_n(-z_1)(-z_2)\cdots(-z_n) = c_0$.

2. **(6 points)** Consider the polynomial

$$p(x) = x^2 - 2\lambda x + 1,$$

   with some parameter $\lambda \in \mathbb{R}$. Which of the following is true?

   A. For any $-1 \leq \lambda \leq 1$ the roots are real numbers.
   B. For any $\lambda \leq -1$ and $\lambda \geq 1$ the roots are real numbers.
   C. For any $-2 \leq \lambda \leq 2$ the roots are real numbers.
   D. For $\lambda > 0$ the roots are positive, and for $\lambda < 0$ the roots are negative.
   E. For $\lambda = \frac{1}{13}$, we have $p(x) > 0$ for all $x \in \mathbb{R}$.
   F. For $\lambda = \frac{1}{13}$, we have $p(x) < 0$ for all $x \in \mathbb{R}$.

3. **(6 points)** Consider the function

$$f(x) = x \sin\left(\frac{1}{x}\right) \quad \text{for } x \neq 0,$$

   and $f(0) := 0$. Which of the following is true?

   A. f is continuous at 0.
   B. f is not continuous at 0.
   C. $|f(x)| \leq |x|$.
   D. $\lim_{x\to 0} f(x) = 0$.
   E. $\lim_{x\to 0} f(x) = 1$.
   F. $\lim_{x\to 0} f(x) = -1$.

4. **(6 points)** What are necessary or sufficient conditions for $f : [a, b] \to \mathbb{R}$ to have a minimum at $c \in (a, b)$?
   A. $f'(x) = 0$ for all $x \in (a, b)$.
   B. $f'(c) = 0$.
   C. $f(c) = 0$.
   D. $f''(c) > 0$.
   E. $f''(c) < 0$.
   F. $f''(c) \geq 0$.

5. **(4 points)** We want to build a fence around a rectangular field. 500 metres of fencing material are available and the field is on one side bounded by a building so that this side won't need any fencing. What is the largest area A that can be fenced in?
   A. None of the given options.
   B. $A = 500m^2$
   C. $A = 50000m^2$
   D. $A = 31250m^2$
   E. $A = 12500m^2$
   F. $A = 25000m^2$

6. **(4 points)** What is the derivative of $f(x) = \sin(e^{2cx^2})$ with respect to x, where $c \in \mathbb{R}$ is some constant?
   A. $f'(x) = e^{2cx^2} \cos(e^{2cx^2})$
   B. $f'(x) = 2cxe^{2cx^2} \sin(e^{2cx^2})$
   C. $f'(x) = 4cxe^{2cx^2} \cos(e^{2cx^2})$
   D. $f'(x) = \cos(4cx^2 e^{2cx^2})$
   E. $f'(x) = cxe^{2cx^2} \cos(e^{2cx^2})$
   F. $f'(x) = xe^{2x^2} \cos(e^{2cx^2})$

7. **(4 points)** Compute the integral

$$\int_0^1 xe^x \, dx.$$

A. e.
B. $2e$.
C. $e-1$.
D. e^x.
E. 2.
F. 1.

8. **(6 points)** Consider the improper integral

$$\int_1^\infty \frac{1}{x^\alpha} \, dx$$

for different parameters $\alpha \in \mathbb{R}$.

A. For $\alpha = 1$ the improper integral is infinite.
B. For $\alpha = 1$ the improper integral is finite and its value is 1.
C. For $\alpha = 1$ the improper integral is finite and its value is $\ln(1+\alpha)$.
D. For $\alpha > 1$ the improper integral is infinite.
E. For $\alpha > 1$ the improper integral is finite and its value is $\frac{1}{\alpha-1}$.
F. For $\alpha > 1$ the improper integral is finite and its value is $\ln \alpha$.

9. **(4 points)** What is the solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3yt$$

with initial condition $y(0) = 1$?

A. $y(t) = e^t$.
B. $y(t) = e^{-t}$.
C. $y(t) = e^{-\frac{3}{2}t^2}$.
D. $y(t) = e^{-t^2}$.
E. $y(t) = e^{-3t}t + 1$.
F. $y(t) = e^{-3t}t$.

10. **(4 points)** What is the scalar product between $u = (2, -3, 1)^T$ and $v = (1, 4, 5)^T$?

A. $u \cdot v = 5$

B. $u \cdot v = -3$

C. $u \cdot v = -5$

D. $u \cdot v = 2$

E. $u \cdot v = -6$

F. $u \cdot v = -10$

11. **(4 points)** Calculate $A^T \cdot B \cdot C$ with

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & -3 \\ 1 & 2 \end{bmatrix}$$

A. $\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$

B. $\begin{bmatrix} 4 & -10 \\ -4 & 10 \end{bmatrix}$

C. $\begin{bmatrix} 4 & 10 \\ 4 & 10 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 4 \\ 4 & -2 \end{bmatrix}$

E. $\begin{bmatrix} 10 & -10 \\ 4 & -4 \end{bmatrix}$

F. $\begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix}$

12. **(6 points)** Which of the following is true?

A. $\{b_1, ..., b_n\}$ with integer n is a basis $\implies$ all b_i, $i = 1, ..., n$ are linearly independent.

B. $u \cdot v = 0 \implies u$ is parallel to v.

C. u is perpendicular to $v \implies u \cdot v$ does not exist.

D. $u \times v = 0 \implies u$ and v are linearly independent.

E. u, v are linearly dependent $\implies au + vb = 0$ only if $a, b = 0$.

F. u, v are linearly independent $\implies au + vb = 0$ only if $a, b = 0$.

13. **(25 points)**

We consider the function

$$f(x) = \frac{e^x}{x-1}.$$

$\boxed{2}$ (a) What is the domain of the function?

$\boxed{3}$ (b) What are the intercepts with the x-axis and with the y-axis?

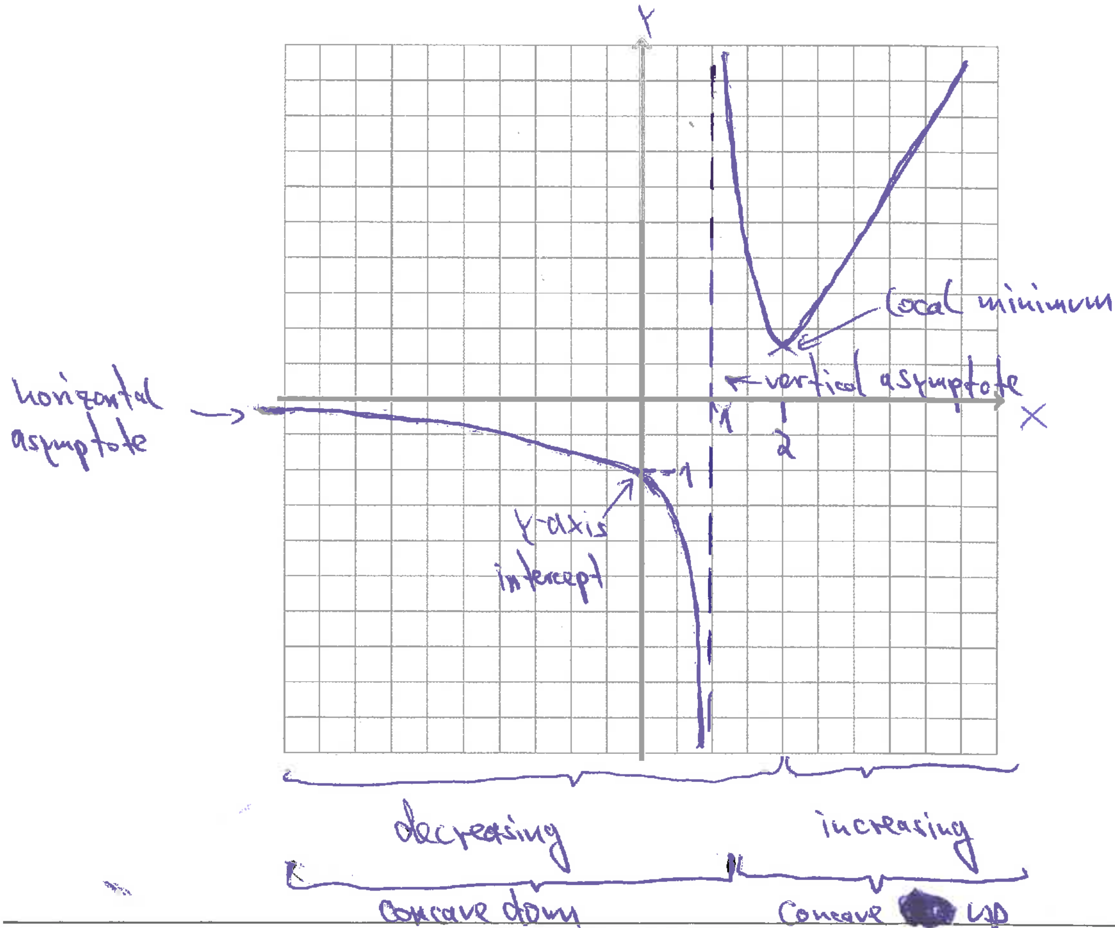
$\boxed{2}$ (c) What are the horizontal asymptotes?

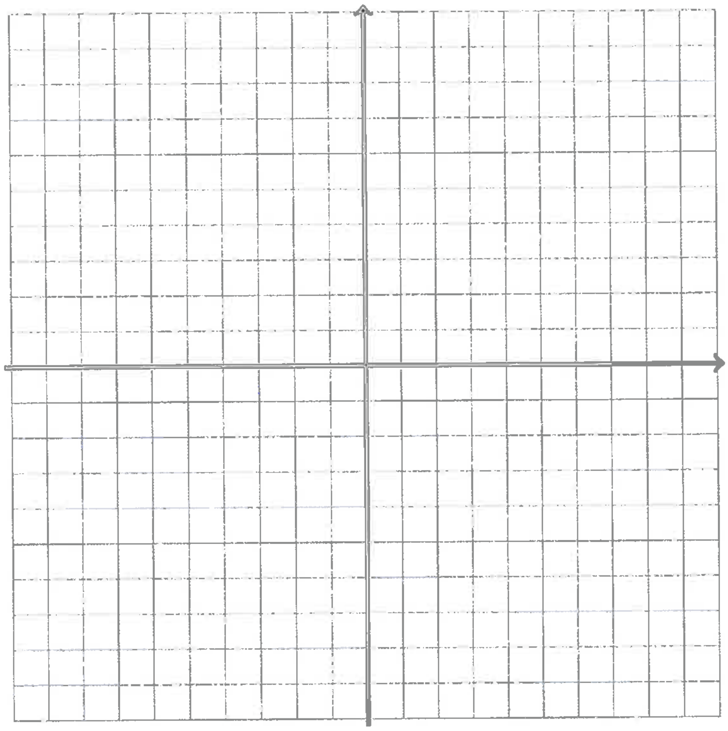
$\boxed{2}$ (d) What are the vertical asymptotes?

$\boxed{6}$ (e) Compute and analyze the first derivative. In which intervals is the function increasing or decreasing, what are the local minima or maxima?

$\boxed{5}$ (f) Compute and analyze the second derivative. In which intervals is the function concave up or concave down, what are the points of inflection?

$\boxed{5}$ (g) Sketch the function. Your drawing needs to include all the qualitative features of the graph discussed in the questions above.





a) The domain is $\mathbb{R}\setminus\{1\} = \{x \in \mathbb{R} : x \neq 1\}$.

b) x-axis: $f(x) = \frac{e^x}{x-1} \overset{!}{=} 0 \Rightarrow e^x \overset{!}{=} 0 \Rightarrow$ no solution

$\Rightarrow$ no intercept with x-axis.

y-axis: $f(0) = \frac{e^0}{-1} = -1$ is the y-axis intercept.

c) Since $\lim\limits_{x\to-\infty} f(x) = 0$, the line $y=0$ is a horizontal asymptote.

d) Since $\lim\limits_{\substack{x\to 1\\ x>0}} f(x) = \infty$ (and $\lim\limits_{\substack{x\to 1\\ x<0}} f(x) = -\infty$), the vertical line $x=1$ is a vertical asymptote.

e) According to the quotient rule:

$$f'(x) = \frac{e^x(x-1) - e^x \cdot 1}{(x-1)^2} = \frac{e^x(x-2)}{(x-1)^2}$$

$f'(x) = 0 \iff x = 2$

$f'(x) < 0$ for $x < 2 \Rightarrow$ here f is decreasing

$f'(x) > 0$ for $x > 2 \Rightarrow$ here f is increasing

$\Rightarrow$ at $x = 2$ there is a local minimum

f) $$f''(x) = \frac{(e^x(x-2) + e^x)(x-1)^2 - e^x(x-2)2(x-1)}{(x-1)^4}$$

$$= \frac{e^x}{(x-1)^3}\left((x-2+1)(x-1) - 2(x-2)\right)$$

$$= e^x \frac{(x^2 - 4x + 5)}{(x-1)^3}$$

Note: the zeros of $x^2 - 4x + 5$ are $x_\pm = 2 \pm \sqrt{-1} = 2 \pm i$

$\Rightarrow$ there are no inflection points. Also:

- for $x < 1$ we have $f''(x) < 0$, so here f is concave down.
- for $x > 1$ we have $f''(x) > 0$, so here f is concave up.

14. **(25 points)**

$\boxed{4}$ (a) For a matrix $A \in M(n \times n)$, give one equivalent statement to "The inverse of A, i.e., a matrix A^{-1} such that $A^{-1}A = \mathbb{1}$, exists".

$\boxed{10}$ (b) Find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$ using row operations as in the lecture.

$\boxed{4}$ (c) Solve the equation $Ax = b$ with $b = (1, 2, 1)^T$.

$\boxed{3}$ (d) What are the values for the Rank and Nullity of A? Briefly explain your answer.

$\boxed{4}$ (e) Now consider the matrix $B = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix}$, i.e., we have changed the last column in A.

What are the values for the Rank and Nullity of B? Briefly explain your answer.

14 a) Multiple solutions are possible:

"A has full rank (or rank n)"

"All rows (or columns) are linearly independent"

"All pivots exist $\neq 0$"

"The determinant of A $\det(A) \neq 0$" (we did not cover this in the lecture)

These are not the only possible solutions.

b)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R3 \Leftarrow R3-2R1]{R2 \Leftarrow R2-R1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -4 & -1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R3 \Leftarrow R3-4R2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -5 & 2 & -4 & 1 \end{array}\right) \xrightarrow[R2 \Leftarrow -R2]{R3 \Leftarrow -\frac{R3}{5}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{5} \end{array}\right)$$

$$\xrightarrow{R2 \Leftarrow R2+R3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{5} \end{array}\right) \xrightarrow{R1 \Leftarrow R1-2R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{5} \end{array}\right)$$

[redacted]

$$A^{-1} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{4}{5} & -\frac{1}{5} \end{pmatrix}.$$

We can check $A^{-1} \cdot A$:

$$\frac{1}{5}\begin{pmatrix} -1 & 2 & 2 \\ 3 & -1 & -1 \\ -2 & 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

c) $Ax = b \Rightarrow x = A^{-1}b$

$$\Rightarrow \frac{1}{5}\begin{pmatrix} -1 & 2 & 2 \\ 3 & -1 & -1 \\ -2 & 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = x$$

(We can check $Ax = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = b \checkmark$)

d) Rank of A is 3 and Nullity is 0. As the matrix is invertible, i.e. A^{-1} exists, rank(A) has to be full, i.e. 3. Rank-Nullity-Theorem tells us, that the nullity has to be 0 as rank(A) + nullity$(A) = 3$

14 e) We can see that the last two columns of B are now linearly dependent. That means they contain "the same information" and therefore the $\underline{\text{rank}(B) = 2}$ rather than 3. Rank-Nullity-Theorem tells us that $\text{rank}(B) - \text{nullity}(B) = 3$

$\Rightarrow$ $\underline{\text{nullity}(B) = 1}$