Week 8: Improper Integrals, ODEs

1. MULTI Single

   The improper integral $\int_{-\infty}^{\infty} x \, \mathrm{d}x$:

   (a) does not exist
   (b) equals ∞
   (c) equals $x^2 + C$
   (d) equals 0

2. MULTI Single

   Compute the improper integral $\int_0^{\infty} e^{-\lambda x} \, \mathrm{d}x$ for any $\lambda > 0$.

   (a) 1
   (b) $\frac{1}{\lambda}$
   (c) The improper integral does not exist.
   (d) λ

3. MULTI Single

   Compute the improper integral $\int_{-\infty}^{\infty} \frac{x}{(x^2+\lambda^2)^2} \, \mathrm{d}x$ for any $\lambda > 0$.

   (a) λ^2
   (b) 0
   (c) 1
   (d) The improper integral does not exist.

4. MULTI Single

   Compute the improper integral $\int_0^1 \frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x$.

   (a) π
   (b) The improper integral does not exist.
   (c) 0
   (d) 1

5. MULTI Single

   Solve $\frac{\mathrm{d}y}{\mathrm{d}t} = y + 1$ with initial condition $y(0) = 1$.

   (a) $y(t) = 1$
   (b) $y(t) = 2e^t - 1$
   (c) $y(t) = 3e^t - 1$
   (d) $y(t) = 3e^t - 2$

6. MULTI Single

   Solve $\frac{\mathrm{d}y}{\mathrm{d}t} = -2yt^2$ with initial condition $y(0) = 2$.

(a) $y(t) = e^{-\frac{t^3}{3}} + 1$
(b) $y(t) = 3e^{-t^3} - 1$
(c) $y(t) = 2e^{-t^3}$
(d) $y(t) = 2e^{-\frac{2t^3}{3}}$

7. MULTI Single

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} - 3e^x = ye^x$. (*Note:* C is a constant to determine the initial condition in the answers below.)

(a) $y = Ce^{3e^x}$
(b) $y = 3^{e^x} - 3 + C$
(c) $y = e^{e^x} - 3 + C$
(d) $y = Ce^{e^x} - 3$

8. MULTI Single

For each of the following equations, determine all the equilibrium points (where $y'(x) = 0$) and classify each as stable (y' changes sign from positive to negative at x or unstable (y' changes sign from negative to positive at x).

- $y_1' = y_1 - y_1^2$
- $y_2' = y_2(y_2 - 1)(y_2 - 2)$
- $y_3' = e^{y_3} - 1$

(a) $y_1 = 1$ (unstable), $y_1 = 0$ (stable), $y_2 = 1$ (unstable), $y_2 = 0, 2$ (stable), $y_3 = 0$ (stable)
(b) $y_1 = 0$ (unstable), $y_1 = 1$ (stable), $y_2 = 0, 2$ (unstable), $y_2 = 1$ (stable), $y_3 = 0$ (stable)
(c) $y_1 = 0$ (unstable), $y_1 = 1$ (stable), $y_2 = 0, 2$ (unstable), $y_2 = 1$ (stable), $y_3 = 0$ (unstable)
(d) $y_1 = 1$ (unstable), $y_1 = 0$ (stable), $y_2 = 1, 2$ (unstable), $y_2 = 0$ (stable), $y_3 = 0$ (stable)

9. MULTI Single

Find the solution to the equation $\hat{a}\psi(x) = 0$, where the action of the operator $\hat{a}$ is given by $\hat{a} := \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)$ and $\psi(x)$ is a function of x.
Remark: This is the ground-state solution to the Quantum Harmonic Oscillator in physics.

(a) $A \tanh x$
(b) $A \cos(b \cdot x)$
(c) $Ae^{-\frac{x^2}{2}}$
(d) Ae^{-x}

10. MULTI Single

The rate of change of the volume of a spherical snowball that is melting is proportional to its area. What is the equation describing the radius of the snowball as a function of time?

(a) $r(t) = r_0 - ct$
(b) $r(t) = r_0 \cdot e^{-ct}$
(c) $r(t) = r_0 - \frac{5}{2}(ct)^{\frac{5}{2}}$
(d) $r(t) = r_0 - \sqrt{ct}$

Total of marks: 10