Week 11: Change of Variables, Differentials, Differential Operators, Optimization

1. MULTI Single

   Let
   $$f(x, y) = \frac{xy}{x^2 + y^2}.$$
   First use the change of coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and then compute $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ when $(x, y) \neq 0$.

   (a) $\partial_r f = 2r, \partial_\theta f = \cos^2(\theta)$
   (b) $\partial_r f = 1, \partial_\theta f = -\sin(2\theta)$
   (c) $\partial_r f = 1, \partial_\theta f = 0$
   (d) $\partial_r f = 0, \partial_\theta f = \cos(2\theta)$

2. MULTI Single

   Determine whether the differential $F = Pdx + Qdy = e^x \sin(y)dx + e^x \cos(y)dx$ is exact. If the differential is inexact what is the difference $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$?

   (a) The differential is inexact and $\partial_y P - \partial_x Q = e^x \sin(y)$
   (b) The differential is exact.
   (c) The differential is inexact and $\partial_y P - \partial_x Q = -2e^x \cos(y)$
   (d) The differential is inexact and $\partial_y P - \partial_x Q = 2e^x \sin(y)$

3. MULTI Single

   Determine whether the differential $F = Pdx + Qdy = (ye^x \sin(y))dx + (e^x + x\cos(y))dy$ is exact. If the differential is inexact what is the difference $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$?

   (a) The differential is inexact and $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = ye^x \sin(y) - e^x - x\cos(y)$.
   (b) The differential is inexact and $\partial_y P - \partial_x Q = y\cos(y)e^x + e^x \sin(y) - \cos(y) - e^x$.
   (c) The differential is inexact and $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = ye^x \sin(y) + x\sin(y)$.
   (d) The differential is exact.

4. MULTI Single

   A force $F = (F_x, F_y)$ written as a differential $F = F_x dx + F_y dy$ is called conservative if there exists V such that $F = -dV$. When an object moves in a closed loop (i.e., the motion has the same start and end points) in a conservative force field, the net work done by the force is zero. Consider a planet moving around the gravitational influence of a star. What is the work done by the force field
   $$F = \frac{kx}{x^2 + y^2}dx + \frac{ky}{x^2 + y^2}$$
   when the planet finishes one revolution? Above, k is a constant.

   (a) The information provided is insufficient to solve the problem.

(b) Work done is zero only when $k = 0$.
(c) Work done is zero.
(d) Work done is non-zero.

5. MULTI Single

Find the curl and the divergence of $F(x, y, z) = (xyz, 0, -x^2y)$.

(a) $\text{curl}(f) = (yz, 0, 0), \text{div}(f) = xz$,
(b) $\text{curl}(f) = (-x^2, 3xy, -xz), \text{div}(f) = yz$,
(c) $\text{curl}(f) = (x^2, xy, xz), \text{div}(f) = 0$,
(d) $\text{curl}(f) = (0, 0, 0), \text{div}(f) = yz$,

6. MULTI Single

Maxwell's equations relating the electric field E and magnetic field H as they vary in time in a region containing no charge and no current can be stated as follows:

$$\text{div}(E) = 0 \qquad \text{div}(H) = 0$$
$$\text{curl}(E) = -\frac{1}{c}\frac{\partial H}{\partial t} \qquad \text{curl}(H) = \frac{1}{c}\frac{\partial E}{\partial t}$$

where c is the speed of light. Compute $\nabla \times (\nabla \times E)$ in terms of E.

(a) Information given is insufficient
(b) $-\left(\frac{1}{c}\frac{\partial E}{\partial t}\right)^2$
(c) $-\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$
(d) $\frac{1}{c}\frac{\partial^2 E}{\partial t^2}$

7. MULTI Single

What are the local maxima, local minima, and saddle points of $f(x, y) = 4 + x^3 + y^3 - 3xy$?

(a) The function has a local minimum at $(1, 1)$. This is the only critical point.
(b) The function has a local minimum at $(0, 0)$ and a local maximum at $(1, 1)$
(c) The function has a saddle point at $(0, 0)$ and a local maximum at $(1, 1)$
(d) The function has a saddle point at $(0, 0)$ and a local minimum at $(1, 1)$.

8. MULTI Single

What are the local maxima, local minima, and saddle points of $f(x, y) = x^3 - 12xy + 8y^3$?

(a) The only critical point at $(0, 0)$ is a saddle point.
(b) The function has local maxima at $(2, 1)$ and at $(0, 0)$.
(c) The function has a local minimum at $(2, 1)$ and a saddle point at $(0, 0)$.
(d) The function has local maximum at $(2, -1)$ and a saddle point at $(0, 0)$.

9. MULTI Single

For functions of one variable it is impossible for a continuous function to have two local maxima and no local minima. However this is not the case for functions of two variables. Let $f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2$. What are all the critical points of f? Which of the critical points are the local maxima?

(a) The only critical point is $(0, 0)$. There is no local maximum.
(b) The only critical points are $(1, 0)$, $(0, 0)$, and $(1, 2)$. Only $(1, 2)$ is a local maximum.
(c) The only critical points $(-1, 0)$ and $(1, 2)$ are local maxima.
(d) The critical points are $(1, 0)$ and $(-1, 2)$. Only $(1, 0)$ is a local maximum.

10. MULTI Single

Find three positive numbers whose sum is 100 and whose product is maximal. What is the product?

(a) $10^6/27$
(b) $(9.7 \times 10^5)/27$
(c) (3.6×10^4)
(d) $(2.2 \times 10^6)/91$

Total of marks: 10