Constructor University February 3, 2025
Fall 2024

Elements of Calculus

Homework 1 (covering Weeks 1 and 2)

Due on February 19, 2025, before the tutorial! Please submit on moodle.

Problem 1 [3 points]

Prove the Pascal triangle property

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Problem 2 [5 points]

Recall the method of proof by induction. Suppose we want to prove that a given statement holds for any $\mathbb{N} \in n \geq n_0$. Then we first show that it holds for $n = n_0$ (usually $n_0 = 0$ or $n_0 = 1$). Then, we assume that the statement holds for some $n \in \mathbb{N}$ ($n \geq n_0$), and we show that it holds for $n+1$. This proves the statement for all $n \geq n_0$.

Use the method of indiction to prove the binomial theorem, i.e., to prove that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

for all $n \in \mathbb{N}$, with the binomial coefficients defined as $\binom{n}{k} := \frac{n!}{k!(n-k)!}$.

Problem 3 [5 points]

Suppose $n+1$ data points (x_0, y_0), $\ldots$, (x_n, y_n) are given. We discussed in class how to find the interpolating polynomial that goes through these data points. Recall that for $n+1$ data points the degree of the polynomial is n or less.

Prove that the interpolating polynomial is unique. *(Hint: Assume f and g are both interpolating polynomials. Then consider the difference of f and g and see what properties it has.)*

Problem 4 [2 points]

Consider the infinite series $\sum_{k=0}^{\infty} (-1)^k a_k$ with $a_k > 0$ for all $k \in \mathbb{N}$. Then the alternating series test (also called Leibniz test) says that if $a_{k+1} \leq a_k$ and $\lim_{k\to\infty} a_k = 0$, then the series converges. Using this test, determine whether the following series converge:

(a) $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$,

(b) $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k+5}}$.

Problem 5 [5 points]

Suppose you would like to compute $\sum_{k=1}^{N} a_k$, and the summands are given as $a_k = b_k - b_{k-1}$ for some other sequence $(b_k)_{k\in\mathbb{N}}$. What is the value of $\sum_{k=1}^{N} a_k$? Use your answer to compute

(a)

$$\sum_{k=1}^{N} \left(k^4 - (k-1)^4\right),$$

(b)

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

Hint: Decompose the summands into partial fractions, i.e., find a, b such that

$$\frac{1}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}.$$