Constructor University
Spring 2025

March 3, 2025

Elements of Calculus

Homework 3 (covering Weeks 5 and 6)

Due on March 18, 2025, before the tutorial! Please submit on moodle.

Problem 1 [2 points]

In class, we discussed a theorem by Lagrange: for continuous functions $f : [a, b] \to \mathbb{R}$, which are differentiable on (a, b), there is an $m \in (a, b)$ such that

$$f(b) - f(a) = f'(m)(b - a).$$

Now, consider the function

$$f(x) = 2x^2 + 3x + 1,$$

and $a = 0$, $b = 1$. Find an m such that the equation above holds.

Problem 2 [3 points]

Find all minima, maxima and points of inflection of $f(x) = \frac{x}{1+x^2}$. Where is the function convex, where is it concave? Based on your results, sketch the graphs of f, f' and f''.

Problem 3 [4 points]

In most applications, using a Taylor series "usually works", meaning that a few terms in the expansion give you a good approximation to the function, and the rest term is small enough. Here, we would like to look at a counter example to this. We consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{, for } x \neq 0 \\ 0 & \text{, for } x = 0. \end{cases}$$

(a) Visualize the function by drawing its graph. (Note: For such an exercise it is important to draw the qualitative features correctly, e.g., behavior at 0 and for large x, and possible maxima/minima.)

(b) Show that the function is continuous at 0.

(c) Compute the derivative $f'(x)$, and evaluate it at $x = 0$. Infer what the values of the higher derivatives $f^{(k)}(0)$ are.

(d) Based on your computations, what is the Talyor series of f around 0? Does the Taylor series of f around 0 converge to f?

Problem 4 [3 points]

Compute the Taylor series of $f(x) = \ln(1+x)$ around $x = 0$, for $-1 < x < 1$. Does the Taylor series also converge for $x = 1$? Does is converge for $x = -1$?

Problem 5 [2 points]

(a) Find the relation between $\sin(ix)$, $\cos(ix)$ and $\sinh(x)$ and $\cosh(x)$.

(b) Prove the identity

$$\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{2}.$$

Problem 6 [3 points]

Let $p(x)$ be a non-constant polynomial of degree less than m with $p(0) \neq 0$. Now consider the function $f(x) = x^m - p(x)$. We would like to approximate the solutions to $f(x) = 0$. Instead of Newton's method, consider the iteration scheme

$$x_{n+1} = [p(x_n)]^{1/m}.$$

(a) Show that if this scheme converges, then the convergence will in general only be of first order.

(b) Consider $m = 3$ and $p(x) = ax^2 + bx + c$. Give an example of non-zero coefficients a, b, c such that for such a $p(x)$ the convergence is actually second order.

Problem 7 [3 points]

We briefly mentioned convex functions in class. This concept can be defined also for functions which are not twice differentiable. We say that a function f is convex in an interval $[a, b]$ if for all $\lambda \in [0, 1]$ and for all $x_1, x_2 \in [a, b]$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2).$$

(a) First, try to understand what this definition means for a convex function f. Draw some function, fix x_1 and x_2, and draw the line $\lambda f(x_1) + (1-\lambda)f(x_2)$ that results from considering all possible $\lambda \in [0, 1]$.

(b) Then, consider a convex function f, some points $x_1, \ldots, x_n$, and some positive $\lambda_1, \ldots, \lambda_n$ with $\sum_{k=1}^{n} \lambda_k = 1$. Prove that

$$f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k f(x_k).$$

This is called Jensen's inequality. (You might see this again next year in your probability class.)