Constructor University | March 17, 2025
Spring 2025

Elements of Calculus

Homework 4 (covering Weeks 7 and 8)

Due on April 1, 2025, before the tutorial! Please submit on moodle.

Problem 1 [5 points]

Suppose we want to compute the length of the curve defined by the continuously differentiable function $f : [a, b] \to \mathbb{R}$. Heuristically speaking, we then need to integrate along the curve, i.e., add up small curve segments $\mathrm{d}s$. For these, Pythagoras gives us $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x$. Indeed, one can show that the curve length L is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x.$$

In this exercise we aim at computing the length of the parabola $f(x) = x^2$ from $a = 0$ to $b = 1$. We find

$$L = \int_0^1 \sqrt{1 + 4x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^2 \sqrt{1 + y^2}\,\mathrm{d}y.$$

(a) Writing $\sqrt{1 + y^2}$ as $1\,\sqrt{1 + y^2}$, use integration by parts in order to express $\int \sqrt{1 + y^2}\,\mathrm{d}y$ in terms of $\int (1 + y^2)^{-1/2}\,\mathrm{d}y$ and some other function.

(b) Then compute

$$\int (1 + y^2)^{-1/2}\,\mathrm{d}y$$

by using the substitution $y = \sinh(x) := \frac{1}{2}\left(e^x - e^{-x}\right)$.

(c) Put parts (a) and (b) together to compute L.

Problem 2 [2 points]

Let f and g be integrable functions. Prove the Cauchy–Schwarz inequality

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq \sqrt{\int_a^b f(x)^2\,dx}\;\sqrt{\int_a^b g(x)^2\,dx}.$$

Hint: Start from the fact that the integral of $\left(f(x) - \lambda g(x)\right)^2$ *is bigger or equal zero for all real* λ.

Problem 3 [4 points]

Let $R(x)$ be a rational function. Then integrals of the form $\int R(\sin x, \cos x, \tan x) dx$ can be solved by using substitution.

(a) One can start by replacing $\sin x = \frac{2y}{1+y^2}$. What is then the substitution for $\cos x$ and $\tan x$?

(b) Use this substitution to calculate

$$\int \frac{1}{2+\sin x} dx.$$

Problem 4 [4 points]

Let $f : [1, \infty) \to [0, \infty)$ be nonincreasing (i.e., for $x \leq y$ we have $f(x) \geq f(y)$).

(a) Show that

$$\sum_{k=2}^{n} f(k) \leq \int_{1}^{n} f(x)\, dx \leq \sum_{k=1}^{n} f(k).$$

Here you can use the fact that nonincreasing functions are integrable.

(b) Show that $\sum_{k=1}^{n} \frac{1}{k}$ diverges logarithmically for large n. (Note: The constant $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right)$ is called Euler-Mascheroni constant.)

(c) Show that $\sum_{k=1}^{\infty} \frac{1}{k^a}$ converges for all $a > 1$.

(d) Does $\sum_{k=2}^{\infty} \frac{1}{k \ln(k)}$ converge or diverge? What about $\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^b}$ for $b > 1$? (*Hint: substitution.*)

Problem 5 [6 points]

(a) The gamma function is defined as

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t}\, dt.$$

Prove that $\Gamma(n) = (n-1)!$ for all natural numbers $n \geq 1$.

(b) In order to calculate integrals of the form $\int_a^b e^{nf(x)}\, dx$ one can use Laplace's method. Assume f has a unique maximum $x_m \in (a, b)$ and that f is twice (continuously) differentiable with $f''(x_m) < 0$. Then,

$$\lim_{n\to\infty} \frac{\int_a^b e^{nf(x)}\, dx}{\sqrt{\frac{2\pi}{n|f''(x_m)|}} e^{nf(x_m)}} = 1,$$

i.e., for very large n,

$$\int_a^b e^{nf(x)}\,dx \approx \sqrt{\frac{2\pi}{n|f''(x_m)|}}e^{nf(x_m)}.$$

Derive the latter formula in a non-rigorous way using a Taylor expansion to second order and just assuming that the remainder term behaves nicely. (You may use the fact that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.)

(c) Use the results from part a) and b) to derive (in a non-rigorous way) Stirling's approximation

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

for large n.