Constructor University
Spring 2025

March 31, 2025

Elements of Calculus

Homework 5 (covering Weeks 9 and 10)

Due on April 23, 2025, before the tutorial! Please submit on moodle.

Problem 1 [2 points]

Consider the function $f(x,y) = e^{-x}y^3$, and the point $\vec{a} = (1,1)$. At $\vec{a}$, in which direction does f increase the most? (If you like, visualize the function, e.g., with geogebra.org/3d, and check that your answer makes sense.)

Problem 2 [6 points]

We consider the function

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

(a) Convince yourself that f is continuous at $(0,0)$. (Choose a sequence (x_n, y_n) that converges to $(0,0)$ and show that $f(x_n, y_n)$ converges to 0.)

(b) Compute the partial derivatives for $(x,y) \neq (0,0)$. Then, using the definition of partial derivatives, compute $(\partial_x f)(0,0)$ and $(\partial_y f)(0,0)$.

(c) Show that f is not differentiable at $(0,0)$ according to the definition of differentiability from class.

Problem 3 [2 points]

Let $h : [0,\infty) \times [0,2\pi) \times [0,\pi] \to \mathbb{R}^3$ be defined as

$$h(r,\varphi,\theta) := (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta).$$

This is the change from spherical to Cartesian coordinates. Compute the Jacobian matrix of h and its determinant.

Problem 4 [5 points]

In class, we discussed the Leibniz rule for computing the derivative of $\int_a^b f(x,t)\mathrm{d}t$. Generalize this rule to the case when the boundaries also depend on x, i.e., find the corresponding rule to compute the derivative of

$$I(x) = \int_{u(x)}^{v(x)} f(x,t)\mathrm{d}t.$$

Apply the new rule to compute the derivative of

$$G(x) = \int_x^{x^2} \frac{\sin(xt)}{t} \mathrm{d}t.$$

Problem 5 [3 points]
Show that the function

$$u : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}, \; x \mapsto \ln \|x\|$$

solves the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x,y) = 0.$$

(Here, $\|x\| = \sqrt{x^2 + y^2}$ is the usual 2-norm.)

Problem 6 [2 points]
Consider the function $f \colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = e^{-y^2} - x^2(y+1)$. Write down the Taylor expansion to second order.