Constructor University
Spring 2025

April 21, 2025

Elements of Calculus

Homework 6 (covering Weeks 11 and 12)

Due on May 6, 2025, before the tutorial! Please submit on moodle.

Problem 1 [**4 points**]

Differentials are useful for approximating small changes in a function. Consider this example (taken from Folland's book). The volume of a right circular cone is given by $V(r,h) = \frac{1}{3}\pi r^2 h$, where r is the base radius and h the height. Consider explicitly $r = 3$ and $h = 5$.

(a) About how much does the volume increase if the height is increased to 5.02 and the radius to 3.01?

(b) If the height is increased to 5.02, by about how much should the radius be decreased to keep the volume constant?

Problem 2 [**4 points**]

Use the method of Lagrange multipliers to find the smallest distance of the curve defined by $x^2 - 2\sqrt{3}xy - y^2 - 2 = 0$ to the origin $(0,0)$. Visualize the situation. *Hint: The distance is given by* $\sqrt{x^2+y^2}$. *But we could as well minimize the function* $f(x,y) = x^2 + y^2$ *(under the constraint above), which makes the computation a bit easier.*

Problem 3 [**4 points**]

Consider a curve in $\mathbb{R}^n$, which we represent by the continuously differentiable function $\vec{g}(t) : [a,b] \to \mathbb{R}^n$ with derivative $\vec{g}\,' \neq 0$. Then

$$\mathrm{d}\vec{g}(t) = \vec{g}\,'(t)\mathrm{d}t.$$

Thus, the length of an infinitesimal curve segment is given by

$$|\mathrm{d}\vec{g}(t)| = |\vec{g}\,'(t)|\mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}g_1}{\mathrm{d}t}\right)^2 + \ldots + \left(\frac{\mathrm{d}g_n}{\mathrm{d}t}\right)^2}\mathrm{d}t.$$

Summing all these up gives us the arc length L of the curve,

$$L = \int_a^b |\vec{g}\,'(t)|\mathrm{d}t,$$

a formula that can also be rigorously proven when g is a C^1 function. Find the lengths of the following curves:

(a) $\vec{g}(t) = (M\cos(t), M\sin(t), Nt)$, $t \in [0, 2\pi]$,

(b) $\vec{g}(t) = (\frac{1}{3}t^3 - t, t^2)$, $t \in [0, 2]$.

But notice that we could as well reparametrize the curve. For any continuously differentiable bijective function $\varphi : [c, d] \to [a, b]$, the function $\vec{g}_\varphi(u) = \vec{g}(\varphi(u))$ would describe the same curve. Check that our formula indeed gives us the same curve length.

Problem 4 [4 points]

Without doing any computation, argue why there must be a minus sign in the following formula:

$$\operatorname{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot (\operatorname{curl}\vec{F}) - \vec{F} \cdot (\operatorname{curl}\vec{G}).$$

Then, prove the formula by direct computation.

Problem 5 [4 points]

In three dimensions, Maxwell's equations for the electric field $\vec{E}$ and the magnetic field $\vec{B}$ in the vacuum read,

$$\operatorname{div}\vec{E} = 0,\ \operatorname{div}\vec{B} = 0,\ \operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t},\ \operatorname{curl}\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t},$$

with some constant $c > 0$.

(a) Show that all component of $\vec{E}$ and $\vec{B}$ satisfy the wave equation

$$\nabla^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$

(b) When f is independent of time, the right-hand side of the wave equation vanishes. Show that $\nabla^2 \frac{1}{|\vec{x}|} = 0$ for $\vec{x} \neq 0$.