Constructor University — May 5, 2025
Spring 2025

Elements of Calculus

Homework 7 (covering Weeks 13 and 14)

This homework cannot be submitted for grading, and solutions will be provided.

Problem 1 [4 points]

Find the general solution to the linear homogeneous ODE

$$y'' + y' - 2y = 0.$$

Then give the solution for the initial condition $y(0) = 2$ and $y'(0) = 5$. What is the behavior of the solution as $x \to \infty$? Also, find one particular solution to the linear inhomogeneous ODE

$$y'' + y' - 2y = e^{-x}.$$

Finally, provide the general solution (i.e., involving two constants) to this inhomogeneous ODE.

Problem 2 [6 points]

We consider the harmonic oscillator in a very general setting, i.e., with friction and a driving force. The corresponding ODE is

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = f(t),$$

where $m, r, k > 0$, and $f(t)$ is an external driving force. We are looking for a solution $x(t)$ of this equation.

(a) Let us consider the homogeneous case first, i.e., $f = 0$.

   (1) Find the general real solution to the homogeneous equation. (You will need to consider different cases depending on m, r, k.)

   (2) How do the solutions behave for large t?

   (3) What is the solution of the harmonic oscillator without friction, i.e., when $r = 0$?

(b) Now consider the inhomogeneous case with periodic force.

(1) Find one particular complex solution to the inhomogeneous equation with $f(t) = e^{i\omega t}$.

(2) By using Euler's formula, use part (1) to find one particular real solution to the inhomogeneous equation with $f(t) = \sin(\omega t)$.

(3) For $f(t) = \sin(\omega t)$ and $r^2 < 2km$, which driving frequency ω makes the amplitude of the solution maximal? This is the so-called resonance frequency. What happens to the amplitude at resonance if the friction goes to zero, i.e., $r \to 0$?

Problem 3 [4 points]

Compute the Fourier transform of the "saw-tooth function" $f(x) = x$ on $[-\pi, \pi)$, periodically extended outside its fundamental domain.

Problem 4 [6 points]

(a) Compute the Fourier series of the function $f(x) = (x - \pi)^2$ for $x \in [0, 2\pi]$ (Example B from Class Session 27).

(b) Finish the proof that we started in class (Session 28) of the fact that the bump function from Example A (Session 27) is mean-square convergent by showing that

$$\sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 = \frac{a}{2\pi},$$

where $\hat{f}_k$ are the Fourier coefficients of the bump function.