Elements of Calculus

Prof. Sören Petrat, Constructor University

Lecture notes from Spring 2025

Session 1
(Week 1 A)

1. Basic Calculus Review

1.1 Sets, Numbers and Polynomials

Topic for Week 1 A: Review of Sets, Numbers, Polynomials, and their Properties

Sets

Let us recall basic set notation. Sets can be defined directly by

- listing their elements, e.g., $A = \{1, 2, 3\}$
- properties that determine the elements, e.g., $B = \{2, 4, 6, 8, \dots\} = \underbrace{\{n \in \mathbb{N} : n \text{ even}\}}_{\text{The set of all natural numbers } n \text{ s.t. } n \text{ is even}}$.

More standard notation:

- $\overbrace{A \subset B}^{A \text{ is a subset of } B}$ $(\Leftrightarrow \underbrace{B \supset A}_{B \text{ is a superset of } A})$ means: $\forall a \in A$ we have $a \in B$.
- $\emptyset := \{\}$ the empty set.
- $A \underbrace{\cup}_{\text{union}} B := \{x : x \in A \text{ or } x \in B\}$. E.g., $\{1,2,3\} \cup \{2,3,4\} = \{1,2,3,4\}$.
- $A \underbrace{\cap}_{\text{intersection}} B := \{x : x \in A \text{ and } x \in B\}$. E.g., $\{1,2,3\} \cap \{2,3,4\} = \{2,3\}$.
- $\underbrace{B \setminus A}_{\text{"B without A"}} = \underbrace{A^c}_{\text{"A complement" (when it is clear which set B is meant)}} = \{x \in B : x \underbrace{\notin}_{\text{not an element of}} A\}$

Numbers

Next we recall the basic number systems.

- Natural numbers: $\mathbb{N} := \{1, 2, 3, 4, \ldots\}$
- Natural numbers with 0: $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$
- Integers: $\mathbb{Z} := -\mathbb{N} \cup \{0\} \cup \mathbb{N}$
- Rational numbers: $\mathbb{Q} := \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\} \right\}$
- Real numbers $\mathbb{R}$. These are a bit harder to define. They "fill the gaps" in the rational numbers by adding to them numbers with infinite and not eventually periodic decimal expansion.

  For example: $\sqrt{2} = 1.41421\ldots \in \mathbb{R}$, but $\sqrt{2} \notin \mathbb{Q}$.
- Complex numbers: $\mathbb{C} := \{ x + iy : x \in \mathbb{R}, y \in \mathbb{R} \}$, with i = imaginary unit def. by the property $i^2 = -1$.

  For $\mathbb{C} \ni z = x + iy$, we call
  - $x = \operatorname{Re} z$ the real part of z
  - $y = \operatorname{Im} z$ the imaginary part of z
  - $\bar{z} = x - iy$ the complex conjugate of z
  - $|z| = \sqrt{\bar{z}z} = \sqrt{x^2 + y^2}$ the absolute value of z

Polynomials

We call $p(x) = \sum_{j=0}^{n} a_j x^j = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ a polynomial with coefficients $a_0, \dots, a_n$.

If $a_n \neq 0$, we call n the degree of the polynomial (the highest power of x with nonzero coefficient).

We call the solutions to $p(x) = 0$ the roots of the polynomial.

Recall the $\boxed{\text{Fundamental Theorem of Algebra}}$:

Every polynomial of degree $n \geq 1$ has n complex roots $z_1, \dots, z_n$ (not necessarily distinct), and can be written in factorized form as $p(x) = a_n (x - z_1)(x - z_2) \cdots (x - z_n)$

Recall also:

- If z is a root of a polynomial with real coefficients, then so is $\bar{z}$.
- The solutions to $x^2 + px + q = 0$ are $x_\pm = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$.

Inequalities and Intervals

less than $\quad$ less than or equal to

Example: For which x does $\underbrace{-3 < \frac{1}{3}(7-2x) \leq 4}_{\text{This means: } -3 < \frac{1}{3}(7-2x) \text{ and } \frac{1}{3}(7-2x) \leq 4.}$ hold?

We "solve" for x: $\quad -9 < 7-2x \leq 12$

$$\Rightarrow -16 < -2x \leq 5$$

$$\Rightarrow -8 < -x \leq \frac{5}{2}$$

if $-8<-x$ then $x+8-8 < x+8-x$, i.e., $x<8$!

$$\Rightarrow -\frac{5}{2} \leq x < 8 \qquad \left(8 > x \geq -\frac{5}{2}\right)$$

In interval notation: $x \in [-\frac{5}{2}, 8)$ $\quad$ (sometimes denoted: $x \in [-\frac{5}{2}, 8[$).

$\frac{5}{2}$ included $\quad$ 8 not included

Example: $x^2 - 4x - 12 > 0$?

We compute the roots: $z_\pm = 2 \pm \sqrt{4+12} = \begin{cases} 6 \\ -2 \end{cases} \Rightarrow x^2-4x-12 = (x+2)(x-6)$.

$\Rightarrow$ Need $\underbrace{x+2>0}_{x>-2}$ and $\underbrace{x-6>0}_{\underline{x>6}}$, or $\underbrace{x+2<0}_{\underline{x<-2}}$ and $\underbrace{x-6<0}_{x<6}$.

$\Rightarrow$ Solution: $x<-2$ or $x>6$.

symbol for "no lower bound"

We can write this as $x \in \underbrace{\mathbb{R} \setminus [-2,6]}_{\mathbb{R} \text{ without the interval } [-2,6]} = (-\infty, -2) \cup (6, \infty)$.

symbol for "no upper bound"

Binomial coefficients

A brief review on binomial coefficients. Goal: compute $(a+b)^n$

We have:

$$
\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= a + b\\
(a+b)^2 &= a^2 + 2ab + b^2\\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3\\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\\
&\vdots
\end{aligned}
$$

Note: the pattern

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1\\
\vdots & & & & & & & & \vdots
\end{array}
$$

is called Pascal's triangle.

The general formula is: $\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \dots \binom{n}{n} b^n}$.

This is called binomial formula (or binomial expansion).

The coefficients $\binom{n}{k}$ or ${}^nC_k$ ("n choose k") are called binomial coefficients.

One can show that $\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$ $\quad (n! = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1)$.