Elements of Calculus

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Lecture notes from Spring 2025

Session 2
(Week 1 B)

1. Basic Calculus Review

1.2 Functions

Topic for Week 1 B: Basics of Functions and their Inverses

Recall the following:

Definition:

A function (or "map") $f: A \to B$ is a rule that assigns to any $x \in A$ exactly one element $y \in B$.

We sometimes write $f: A \to B, x \mapsto f(x)$.

Furthermore:
- A is called **domain**, B is called codomain.
- **Range(f)** $:= \{ f(x) : x \in A \} \subset B$.
- The **graph of f** is $:= \{ (x,y) : x \in A, y = f(x) \}$ (i.e. the set of all points s.t. $y = f(x)$).

Next: Standard functions we need to know.

- **Absolute value.** $\text{abs}: \mathbb{R} \to [0,\infty), x \mapsto |x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$
- **Linear function.** $f_a: \mathbb{R} \to \mathbb{R}, x \mapsto ax$ (for $a \neq 0$)
- **Parabola.** $f: \mathbb{R} \to [0,\infty), x \mapsto x^2$

- **Hyperbola.** $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\},\ x \mapsto \frac{1}{x}$
- **Trigonometric functions:** $\sin, \cos: \mathbb{R} \to [-1,1]$, $\tan: \mathbb{R}\setminus\{\frac{\pi}{2} + n\pi : n \in \mathbb{Z}\} \to \mathbb{R}$ $\left(\tan x = \frac{\sin x}{\cos x}\right)$

- **Exponential function.**

For $n \in \mathbb{N}$, $\mathbb{R} \ni a > 0$ we have $a^n := \underbrace{a \cdot a \cdots a}_{n \text{ times}}$.

It satisfies $a^{n+m} = a^n a^m$. $(*)$

One can now define a^x for $x \in \mathbb{Z}, x \in \mathbb{Q}$ while keeping $(*)$ true:

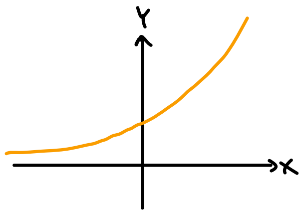
- For $n \in \mathbb{N}$: $1 = a^0 = a^{n-n} = a^n a^{-n} \Rightarrow a^{-n} := \frac{1}{a^n}$.
- For $q \in \mathbb{Z}, q \neq 0$: $a = a^{\frac{q}{q}} = \left(a^{\frac{1}{q}}\right)^q \Rightarrow a^{\frac{1}{q}} = \sqrt[q]{a}$ (the q-th root of a)

  $\Rightarrow$ for $r = \frac{p}{q}$, $p \in \mathbb{Q}, q \in \mathbb{Q}\setminus\{0\}$: $a^r = (\sqrt[q]{a})^p$

Result: a^x can be defined for $a > 0, x \in \mathbb{Q}$, and satisfies $a^{x+y} = a^x a^y$.

Note: We also have $(a^x)^y = a^{xy} = (a^y)^x$ $\left(\neq a^{(x^y)}\right)$

Graph of a^x:



We def. a^x for $x \in \mathbb{R}$ later.

Also recall:

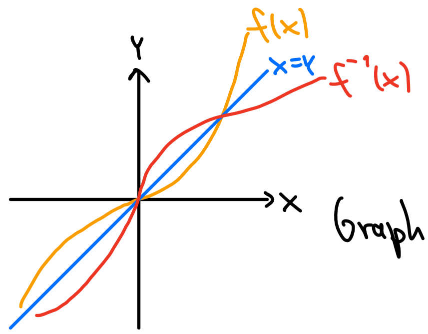
Definition:

If $f: A \to B$, then $g: B \to A$ is called the **inverse of f** if

- $g(f(x)) = x \quad \forall x \in A$, $(*)$
- $f(g(y)) = y \quad \forall y \in B$. $(**)$

Note: g is usually denoted by f^{-1}. ← this does not mean $\frac{1}{f}$ here!

Graphically:



Graph of $y=f^{-1}(x) \iff \underbrace{f(y)=x}_{\text{Same as } f(x)=y \text{ but with } x \text{ and } y \text{ interchanged.}}$.

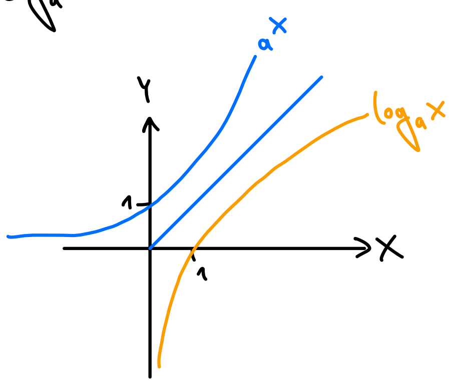
$\Rightarrow$ "Graph of $y=f^{-1}(x)$" = "Graph of $y=f(x)$ reflected at $y=x$ (blue line)".

Example: Logarithm

For $f(x)=a^x$, $a>0$, we call the inverse the logarithm to base a: $\boxed{\log_a y.}$

$\Rightarrow \log_a a^x = x$.

Graph:



Note: $\log_a 1 = 0$ (bc. $a^0=1$).

We have $\boxed{\log_a(xy)=\log_a x+\log_a y}$.

↳ Why? $a^{\log_a(xy)} = xy = a^{\log_a x}\, a^{\log_a y} \overset{\text{by } (*)}{=} a^{\log_a x+\log_a y}$

One more definition:

<u>Definition:</u> Let $f: A \to B$.

- f is called $\boxed{\text{injective}}$ (or "one-to-one") if $\forall x, x' \in A$ we have: $f(x) = f(x') \Rightarrow x = x'$
- f is called $\boxed{\text{surjective}}$ (or "onto") if $\forall y \in B \; \exists x \in A$ s.t. $f(x) = y$.
- f is called $\boxed{\text{bijective}}$ if it is injective and surjective.

Note: By this definition, the bijective functions are exactly the invertible ones.

Examples:

- $f(x) = x^3$ is injective
- $f(x) = x^2$ is not injective $\left((-1)^2 = 1 = 1^2\right.$, i.e., two different x's have the same $f(x)$).
- $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is not surjective (if $y = -1$, $\nexists\, x \in \mathbb{R}$ s.t. $f(x) = -1$).
- $f: \mathbb{R} \to [0,\infty), x \mapsto x^2$ is surjective ($\forall y \in [0,\infty) \; \exists x \in \mathbb{R}$ s.t. $f(x) = y$).
- $f: \mathbb{R} \to [0,\infty), x \mapsto x^2$ is not bijective, but $f: [0,\infty) \to [0,\infty), x \mapsto x^2$ is bijective
- $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^3$ is bijective.