Elements of Calculus

Example Session Week 1

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Lecture notes from Spring 2025

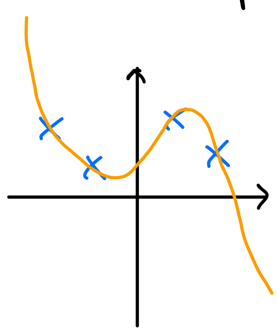
Example Session for:

Week 1 A: Review of Sets, Numbers, Polynomials, and their Properties

Week 1 B: Basics of Functions and their Inverses

Polynomial Interpolation

Goal: Find a polynomial of degree n or less that goes through given points $(x_0,y_0), \ldots, (x_n,y_n)$.



$P(x) = \sum_{j=0}^{n} a_j x^j$, and we need to find the right coefficients $a_0, a_1, \ldots, a_n$.

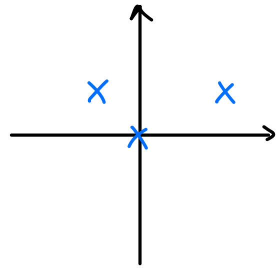
Plugging in the given points leads to:

$$\begin{aligned} a_n x_0^n + a_{n-1} x_0^{n-1} + \ldots + a_0 &= y_0 \\ a_n x_1^n + a_{n-1} x_1^{n-1} + \ldots + a_0 &= y_1 \\ &\vdots \\ a_n x_n^n + a_{n-1} x_n^{n-1} + \ldots + a_0 &= y_n \end{aligned}$$

$\Longrightarrow$ A system of $n+1$ linear equations for the $n+1$ unknowns $a_0, a_1, \ldots, a_n$.

This can be solved by Gaussian elimination.

Example: Find the parabola going through $(-1,1)$, $(0,0)$, and $(2,1)$.

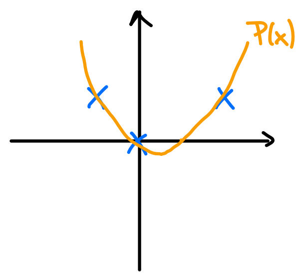


$$\Rightarrow \text{Need to solve} \quad \begin{aligned} a_2(-1)^2 + a_1(-1) + a_0 &= 1 \\ a_2 0^2 + a_1 0 + a_0 &= 0 \\ a_2 2^2 + a_1 2 + a_0 &= 1 \end{aligned}$$

$$\Rightarrow a_0 = 0, \text{ and it remains to solve} \quad \begin{aligned} a_2 - a_1 &= 1 \\ 4a_2 + 2a_1 &= 1 \end{aligned}$$

$$\xRightarrow{-4R1+R2 \to R2} \quad \begin{aligned} a_2 - a_1 &= 1 \\ 6a_1 &= -3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a_2 &= \tfrac{1}{2} \\ a_1 &= -\tfrac{1}{2} \end{aligned}$$

$\Rightarrow$ The parabola is $P(x) = \frac{1}{2}x^2 - \frac{1}{2}x$.



Roots of Polynomials of Degree $n>2$

Formulas for $n=3,4$ complicated, general formulas for $n \geq 5$ don't exist.

$\Rightarrow$ Need numerical techniques such as Newton's method (later).

Often, one root can be guessed, e.g. $x^3 - 2x^2 - 5x + 6$ has a root 1, since $1-2-5+6=0$.

$$\Rightarrow x^3 - 2x^2 - 5x + 6 = (x-1)\underbrace{(x^2 + ax + b)}_{\text{some polynomial with unknown } a,b}.$$

But now we can compare coefficients:

$$x^3-2x^2-5x+6 = x^3+(a-1)x^2+(b-a)x-b \qquad \Rightarrow a=-1, b=-6$$

$b=-6$

$a-1=-2=-2$ ✓ $\qquad$ $b-a=-6-a=-5 \Rightarrow a=-1$

Now: roots of x^2-x-6 are $x_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4}+6} = \frac{1}{2} \pm \sqrt{\frac{25}{4}} = \frac{1}{2} \pm \frac{5}{2} = \begin{cases} 3 & \text{for } + \\ -2 & \text{for } - \end{cases}$

$$\Rightarrow x^3-2x^2-5x+6 = (x-1)(x-3)(x+2).$$

Proof by Induction

Proof Method: Induction

We aim at proving a claim for all natural numbers $n \geq n_0 \in \mathbb{N}_0$.

Step 1: Show that the claim holds for $n = n_0$. (Usually easy, since usually $n_0 = 0$ or 1.)

Step 2: We assume the claim holds for some $n \in \mathbb{N}$ and then prove that it holds for $n+1$.

By step 1 the claim holds for n_0, by step 2 it holds for n_0+1, by step 2 again it holds for n_0+2, and so on $\Rightarrow$ Claim holds for all $n \geq n_0$.

Example: Claim: $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$.

Step 1: $n=1$: $\sum_{k=0}^{1} k = 1 = \frac{1(1+1)}{2}$ ✓

Step 2: $\sum_{k=0}^{n+1} k = \underbrace{\sum_{k=0}^{n} k}_{= \frac{n(n+1)}{2} \text{ by induction assumption}} + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2}+1\right) = \frac{(n+1)(n+1+1)}{2}$, which is the desired formula for $n+1$.